Some results on Graceful graphs and Latin Transversals

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Overview

1. Graceful graphs
2. Lopsided Lovàsz Local Lemma
3. Latin Transversals

Definition (Graceful labeling)

A *graceful labeling* of a graph $G = (V, E)$ is a vertex labeling $f : V \to [0, m]$, where $m = |E|$, such that f is injective and the induced labeling on the edges $f : E \to [1, m]$ defined by $f(u, v) = |f(u) - f(v)|$ is also injective.

Definition (Graceful graph)

If the graph G admits a graceful labeling, we say that G is *graceful*.

Almost all graphs are not graceful

Theorem (Erdős unpublished, Grahame and Sloane (1980))

Almost all graphs are not graceful.

Proof.

First note that there are $\binom{\binom{n}{2}}{m}$ graphs with n vertices and m edges. Let f be a vertex labeling on n vertices with distinct number from $[0, m]$. There are $(m+1)m \cdots (m-n) \leq (m+1)^n$ such labelings.

Almost all graphs are not graceful

Let us count how many graphs there are for which f is a graceful labeling. Let p_i be the number of pairs of vertices $\{u, v\}$ such that $|f(u) - f(v)| = i$. Clearly, $\sum_i p_i = \binom{n}{2}$. A graph is graceful with the f-labeling if we take one edge from each class counted by p_i. Thus there are

$$\prod_{i=1}^{m} p_i \leq \left(\frac{n(n-1)}{2m}\right)^m$$

graphs for which f is a graceful labeling. This product is maximized when all the p_i's are equal.

Almost all graphs are not graceful

Therefore there are at most

$$(m+1)^n \left(\frac{n(n-1)}{2m}\right)^m$$

graceful graphs. Finally, we show that the ratio

$$\rho = \frac{(m+1)^n \left(\frac{n(n-1)}{2m}\right)^m}{\binom{\binom{n}{2}}{m}}$$

tends to 0 as $n \to \infty$.

Almost all graphs are not graceful

Writing $m = (1/2 - \mu)\binom{n}{2}$ with $\mu \in (-1/2, 1/2)$. We have

$$\rho < \frac{(m+1)^n \sqrt{8\binom{n}{2}(\frac{1}{2}-\mu)(\frac{1}{2}+\mu)}}{(\frac{1}{2}-\mu)^m 2^{\binom{n}{2}h(\frac{1}{2}-\mu)}}$$

where $h(x) = -x\log_2 x - (1-x)\log_2(1-x)$. Simplifying the denominator

$$\rho < \frac{(m+1)^n \sqrt{8\binom{n}{2}(\frac{1}{2}-\mu)(\frac{1}{2}+\mu)}}{2^{-\binom{n}{2}(\frac{1}{2}+\mu)\log_2(\frac{1}{2}+\mu)}}$$

taking the logarithm on both sides it is easy to see that the RHS tends to $-\infty$ as $n \to \infty$. Then $\rho \to 0$ as $n \to \infty$.

Lemma (Lopsided Local Lemma - Symmetric case)

Let $A_1, A_2, \ldots, A_n$ be events in an arbitrary probability space. A graph $G = (V, E)$ on the set of vertices $V = \{1, 2, \ldots, n\}$ is called lopsidedependency graph for the A_i's if

$$\Pr(A_i | \cap_{j \in S} \overline{A}_j) \leq \Pr(A_i)$$

for all i, S with $i \notin S$ and no $j \in S$ adjacent to i in G. Suppose that all events have probability at most p and that each vertex in G has degree at most d. If

$$ep(d+1) \leq 1$$

then $\Pr(\cap_{i=1}^{n} \overline{A}_i) > 0$.

Definition (Latin Transversal)

Let $A = (a_{ij})$ be a $n \times n$ matrix with integer entries. A permutation π is called a *Latin transversal* if the entries $a_{i\pi(i)}$ for $i = 1, \dots, n$ are all different.

Example

$$A = \begin{pmatrix} 1 & 2 & 3 & \mathbf{4} \\ \mathbf{3} & 1 & 4 & 5 \\ 2 & 5 & \mathbf{1} & 2 \\ 4 & \mathbf{2} & 3 & 5 \end{pmatrix}, \quad \pi = (4, 1, 3, 2)$$

Theorem (Existence of Latin Transversals)

Let $A = (a_{ij})$ be a $n \times n$ matrix with integer entries. Suppose $k \leq \frac{n-1}{4e}$ and suppose no integer appears in more than k entries of A. Then A has a Latin Transversal.

Proof.

Let π be a random permutation of $\{1, 2, \ldots, n\}$ chosing with uniform distribution among all $n!$ permutations. Denote by T the set of all (i, j, i', j') such that $i < i'$, $j \neq j'$ and $a_{ij} = a_{i'j'}$. For each $(i, j, i', j') \in T$ let $A_{iji'j'}$ be the event that $\pi(i) = j$, $\pi(i') = j'$. Clearly $\Pr(A_{iji'j'}) = \frac{1}{n(n-1)}$.

Latin Transversals

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⇓

If none of these events hold with positive probability then a **Latin Transversal** exists.

Let G be a symmetric graph on the vertex set T and (i, j, i', j') is adjacent to (p, q, p', q') iff $\{i, i'\} \cap \{p, p'\} \neq \emptyset$ or $\{j, j'\} \cap \{q, q'\} \neq \emptyset$. The maximum degree of G is less than $4nk$. In fact there are at most $4n$ choices of (s, t) with either $s \in \{i, i'\}$ or $t \in \{j, j'\}$ and for each of these choices there are less than k choices for $(s', t') \neq (s, t)$ and $a_{st} = a_{s't'}$. By hypothesis we have $e \cdot 4nk \cdot \frac{1}{n(n-1)} \leq 1$ and so, by the Lopsided Local Lemma we only need to prove that

$$\Pr(A_{iji'j'} | \cap_S \overline{A}_{pqp'q'}) \leq \frac{1}{n(n-1)}$$

for any $(i, j, i', j') \in T$ and any set S of members of T nonadjacent in G to (i, j, i', j').

By symmetry, assume $i = j = 1$, $i' = j' = 2$ and hence none of the p's or q's are equal to 1 or 2. We say that π is *good* if it satisfies $\cap_S \overline{A}_{pqp'q'}$. Let S_{kl} denote the set of all good permutations π such that $\pi(1) = k$ and $\pi(2) = l$.

Claim. $|S_{12}| \leq |S_{kl}|$ for all $k \neq l$.
Suppose $k, l > 2$. For each $\pi \in S_{12}$, where $\pi(x) = k$ and $\pi(y) = l$, define π^* such that $\pi^*(1) = k$, $\pi^*(2) = l$, $\pi^*(x) = 1$, $\pi^*(y) = 2$ and $\pi^*(t) = \pi(t)$ for all $t \neq 1, 2, x, y$. Thus $\pi^* \in S_{kl}$.

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⇓

The mapping $\pi \in S_{12} \to \pi^* \in S_{kl}$ is injective.
Then $|S_{12}| \leq |S_{kl}|$.

It follows that

$$\Pr(A_{1122}| \cap_S \overline{A}_{pqp'q'}) = \frac{|S_{12}|}{\sum_{k \neq l} |S_{kl}|}.$$

Since $|S_{kl}| \geq |S_{12}|$ for all $k \neq l$ then

$$\Pr(A_{1122}| \cap_S \overline{A}_{pqp'q'}) \leq \frac{1}{n(n-1)}.$$

Therefore, by symmetry and applying the Lopsided Local Lemma the Theorem follows. □

References

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