In[•]:=
```
(* ADDITIONAL MATERIAL
 "A NEW CLASS OF PARTIALLY FILLED ARRAYS AND ITS APPLICATIONS" *)
(* Simone Costa, Stefano Della Fiore, and Anita Pasotti *)

(* We prove that the partial sums for each of the first
  k-1 columns of the non zero sum Heffter
  array A defined in Theorem 5.2 are pairwise distinct *)

(* In all the document we suppose that n ≥ k ≥ 1 and 1 ≤ i ≤ k-1 *)
```

In[•]:=
```
(* CASE 1) k even and i odd (k odd and i even) *)

(* We first parametrize each partial sum as follows *)
f1[a_] := -a (k - 1) - k (n - k + 1 + i);
f2[b_] := b (k - 1);
f3[g_] := -((k - i - 1) / 2) (k - 1) - k (n - k + 1 + i) + g (k - 1) + i;
f4[d_] := -((k - i - 1) / 2) (k - 1) - k (n - k + 1 + i) - d (k - 1);

Style["Parametric forms for k even and i odd", FontWeight → Bold]
Print["1st parametric form: ", TraditionalForm[f1[α]],
  " for ", TraditionalForm[0 ≤ α ≤ (k - i - 1) / 2]];
Print["2nd parametric form: ", TraditionalForm[f2[β]],
  " for ", TraditionalForm[1 ≤ β ≤ (k - i - 1) / 2]];
Print["3rd parametric form: ", TraditionalForm[f3[γ]],
  " for ", TraditionalForm[0 ≤ γ ≤ (i - 1) / 2]];
Print["4th parametric form: ", TraditionalForm[f4[δ]],
  " for ", TraditionalForm[1 ≤ δ ≤ (i - 1) / 2]];

Style["Parametric forms for k odd and i even", FontWeight → Bold]
Print["1st parametric form: ", TraditionalForm[-f1[α]],
  " for ", TraditionalForm[0 ≤ α ≤ (k - i - 1) / 2]];
Print["2nd parametric form: ", TraditionalForm[-f2[β]],
  " for ", TraditionalForm[1 ≤ β ≤ (k - i - 1) / 2]];
Print["3rd parametric form: ", TraditionalForm[-f3[γ]],
  " for ", TraditionalForm[0 ≤ γ ≤ i / 2 - 1]];
Print["4th parametric form: ", TraditionalForm[-f4[δ]],
  " for ", TraditionalForm[1 ≤ δ ≤ i / 2]];
```

Out[•]= **Parametric forms for k even and i odd**

1st parametric form: $-k\left(i-k+n+1\right)-\alpha\left(k-1\right)$ for $0\leq\alpha\leq\frac{1}{2}\left(-i+k-1\right)$

2nd parametric form: $\beta\left(k-1\right)$ for $1\leq\beta\leq\frac{1}{2}\left(-i+k-1\right)$

3rd parametric form: $-k\left(i-k+n+1\right)+\frac{1}{2}\left(k-1\right)\left(i-k+1\right)+i+\gamma\left(k-1\right)$ for $0\leq\gamma\leq\frac{i-1}{2}$

4th parametric form: $-k\left(i-k+n+1\right)+\frac{1}{2}\left(k-1\right)\left(i-k+1\right)-\delta\left(k-1\right)$ for $1\leq\delta\leq\frac{i-1}{2}$

Out[ ]= **Parametric forms for k odd and i even**

1st parametric form: $k\left(i-k+n+1\right)+\alpha\left(k-1\right)$ for $0\leq\alpha\leq\frac{1}{2}\left(-i+k-1\right)$

2nd parametric form: $\beta\left(-\left(k-1\right)\right)$ for $1\leq\beta\leq\frac{1}{2}\left(-i+k-1\right)$

3rd parametric form: $k\left(i-k+n+1\right)-\frac{1}{2}\left(k-1\right)\left(i-k+1\right)-i-\gamma\left(k-1\right)$ for $0\leq\gamma\leq\frac{i}{2}-1$

4th parametric form: $k\left(i-k+n+1\right)-\frac{1}{2}\left(k-1\right)\left(i-k+1\right)+\delta\left(k-1\right)$ for $1\leq\delta\leq\frac{i}{2}$

In[ ]:=
```
(* Comparison 1st - 1st *)
Print["-2nk-1 < ", TraditionalForm[Simplify[f1[α1] - f1[α2]]],
  " < 2nk+1 and ", TraditionalForm[Simplify[f1[α1] - f1[α2]] ≠ 0],
  " since ", TraditionalForm[α1 ≠ α2]];
```

-2nk-1 < $-\left(k-1\right)\left(\alpha 1-\alpha 2\right)$ < 2nk+1 and $-\left(k-1\right)\left(\alpha 1-\alpha 2\right)\neq 0$ since $\alpha 1\neq\alpha 2$

In[ ]:=
```
(* Comparison 1st - 2nd *)
max = Maximize[{f2[b] - f1[a], i ≥ 1, i ≤ k - 1,
    a == (k - i - 1) / 2, b == (k - i - 1) / 2, k ≥ 1, n ≥ k}, {a, b, i}];
max = max[[1]][[1]][[2]][[1]];
min =
  Minimize[{f2[b] - f1[a], i ≥ 1, i ≤ k - 1, a == 0, b == 1, k ≥ 1, n ≥ k}, {i, b, a}];
min = min[[1]][[1]][[2]][[1]];
Print["0 < ", TraditionalForm[Simplify[min]],
  " ≤ ", TraditionalForm[Simplify[f2[β] - f1[α]]],
  " ≤ ", TraditionalForm[Simplify[max]], " < 2nk+1"];
```

0 < $-k^2+k\,(n+3)-1$ ≤ $-\alpha-\beta+k\left(\alpha+\beta+i+n+1\right)-k^2$ ≤ $k\,n$ < 2nk+1

In[•]:=
```
(* Comparison 1st - 3rd *)
max = Maximize[{f3[g] - f1[a], i ≥ 1, i ≤ k - 1, a ≥ 0,
    a ≤ (k - i - 1) / 2, g ≥ 0, g ≤ (i - 1) / 2, k ≥ 1, n ≥ k}, {i, g, a}];
max = max[[1]][[1]][[2]][[1]];
min = Minimize[{f3[g] - f1[a], i ≥ 1, i ≤ k - 1, a ≥ 0,
    a ≤ (k - i - 1) / 2, g ≥ 0, g ≤ (i - 1) / 2, k ≥ 1, n ≥ k}, {i, g, a}];
min = min[[1]][[1]][[2]][[1]];
sol = Reduce[{f1[a] - f3[g] == 0 && i == k - 1 && a ≥ 0 && a ≤ (k - i - 1) / 2 &&
     g ≥ 0 && g ≤ (i - 1) / 2 && k ≥ 1 && n ≥ k }, {k, n, i, g, a}, Integers];
Print["-2nk-1 < ", TraditionalForm[Simplify[min]], " ≤ ",
  TraditionalForm[(k - 1) (α + γ - 1/2 (k - i - 1)) + i ],
  " ≤ ", TraditionalForm[Simplify[max]], " < 2nk+1" ];
Print["Since ", TraditionalForm[(k - 1) (α + γ - 1/2 (k - i - 1)) + i ],
  " = 0 if and only if ", Congruent[i, 0],
  "(mod k-1) then the only possible value for i is k-1 because ",
  TraditionalForm[1 ≤ i ≤ k - 1], "."];
Print["Does ", TraditionalForm[(k - 1) (+α + γ + 1) ],
  " = 0 have a solution? ", sol, "."];
```

$-2nk-1 < -\frac{1}{2}(k-3)\,k \leq (k-1)\left(\alpha+\gamma+\frac{1}{2}(i-k+1)\right)+i \leq \frac{1}{2}(k-1)\,k < 2nk+1$

Since $(k-1)\left(\alpha+\gamma+\frac{1}{2}(i-k+1)\right)+i = 0$ if and only if $i \equiv 0$ (mod k-1) then the only possible value for i is k-1 because $1 \leq i \leq k-1$.

Does $(k-1)(\alpha+\gamma+1) = 0$ have a solution? False.

In[•]:=
```
(* Comparison 1st - 4th *)
max = Maximize[{f1[a] - f4[d], i ≥ 1, i ≤ k - 1, a ≥ 0,
    a ≤ (k - i - 1) / 2, d ≥ 1, d ≤ i / 2, k ≥ 1, n ≥ k}, {a, d, i}];
max = max[[1]][[1]][[2]][[1]];
min = Minimize[{f1[a] - f4[d], i ≥ 1, i ≤ k - 1, a ≥ 0,
    a ≤ (k - i - 1) / 2, d ≥ 1, d ≤ i / 2, k ≥ 1, n ≥ k}, {a, d, i}];
min = min[[1]][[1]][[2]][[1]];
Print["0 < ", TraditionalForm[Simplify[min]],
  " ≤ ", TraditionalForm[Simplify[f1[α] - f4[δ]]],
  " ≤ ", TraditionalForm[Simplify[max]], " < 2nk+1" ];
```

$0 < k-1 \leq \frac{1}{2}(k-1)\,(-2\,\alpha+2\,\delta-i+k-1) \leq \frac{1}{2}(k-1)^2 < 2nk+1$

In[•]:=
```
(* Comparison 2nd - 2nd *)
Print["-2nk-1 < ", TraditionalForm[Simplify[f2[β1] - f2[β2]]],
  " < 2nk+1 and ", TraditionalForm[Simplify[f2[β1] - f2[β2]] ≠ 0],
  " since ", TraditionalForm[β1 ≠ β2]];
```

$-2nk-1 < (k-1)\,(\beta 1-\beta 2) < 2nk+1$ and $(k-1)\,(\beta 1-\beta 2) \neq 0$ since $\beta 1 \neq \beta 2$

In[•]:=
```
(* Comparison 2nd - 3rd *)
max = Maximize[{f2[b] - f3[g], i ≥ 1, i ≤ k - 1, b ≥ 1,
    b ≤ (k - i - 1) / 2, g ≥ 0, g ≤ (i - 1) / 2, k ≥ 1, n ≥ k}, {b, g, i}];
max = max[[1]][[1]][[2]][[1]];
min = Minimize[{f2[b] - f3[g], i ≥ 1, i ≤ k - 1, b ≥ 1,
    b ≤ (k - i - 1) / 2, g ≥ 0, g ≤ (i - 1) / 2, k ≥ 1, n ≥ k}, {b, g, i}];
min = min[[1]][[1]][[2]][[1]];
Print["0 < ", TraditionalForm[Simplify[min]],
  " ≤ ", TraditionalForm[Simplify[f2[β] - f3[γ]]],
  " ≤ ", TraditionalForm[Simplify[max]], " < 2nk+1" ];
```

$$0 < -\frac{k^2}{2} + k\left(n + \frac{3}{2}\right) - 1 \le \frac{1}{2}\left(-2\beta + 2\gamma + i\left(k-1\right) - k^2 + 2k(\beta - \gamma + n) + 1\right) \le k(n-1) + 1 < \text{2nk+1}$$

In[•]:=
```
(* Comparison 2nd - 4th *)
max = Maximize[{f2[b] - f4[d], i ≥ 1,
    i ≤ k - 1, b == (k - i - 1) / 2, d == i / 2, k ≥ 1, n ≥ k}, {b, d, i}];
max = max[[1]][[1]][[2]][[1]];
min =
  Minimize[{f2[b] - f4[d], i ≥ 1, i ≤ k - 1, b == 1, d == 1, k ≥ 1, n ≥ k}, {b, d, i}];
min = min[[1]][[1]][[2]][[1]];
Print["0 < ", TraditionalForm[Simplify[min]],
  " ≤ ", TraditionalForm[Simplify[f2[β] - f4[δ]]],
  " ≤ ", TraditionalForm[Simplify[max]], " < 2nk+1"];
```

$$0 < -\frac{k^2}{2} + k\left(n + \frac{5}{2}\right) - 1 \le$$

$$\frac{1}{2}\left(-2\beta - 2\delta + i\left(k+1\right) - k^2 + 2k(\beta + \delta + n) + 1\right) \le \frac{1}{2}\left(k^2 + 2k(n-1) + 1\right) < \text{2nk+1}$$

In[•]:=
```
(* Comparison 3rd - 3rd *)
Print["-2nk-1 < ", TraditionalForm[Simplify[f3[γ1] - f3[γ2]]],
  " < 2nk+1 and ", TraditionalForm[Simplify[f3[γ1] - f3[γ2]] ≠ 0],
  " since ", TraditionalForm[γ1 ≠ γ2]];
```

$$\text{-2nk-1} < \left(k-1\right)(\gamma1 - \gamma2) < \text{2nk+1 and } \left(k-1\right)(\gamma1 - \gamma2) \ne 0 \text{ since } \gamma1 \ne \gamma2$$

In[•]:=
```
(* Comparison 3rd - 4th *)
max = Maximize[{f3[g] - f4[d], i ≥ 1,
    i ≤ k - 1, d == i / 2, g == (i - 1) / 2, k ≥ 1, n ≥ k}, {d, g, i}];
max = max[[1]][[1]][[2]][[1]];
min =
  Minimize[{f3[g] - f4[d], i ≥ 1, i ≤ k - 1, d == 1, g == 0, k ≥ 1, n ≥ k}, {d, g, i}];
min = min[[1]][[1]][[2]][[1]];
Print["0 < ", TraditionalForm[Simplify[min]],
  " ≤ ", TraditionalForm[Simplify[f2[γ] - f4[δ]]],
  " ≤ ", TraditionalForm[Simplify[max]], " < 2nk+1"];
```

$$0 < k \le \frac{1}{2}\left(-2\gamma - 2\delta + i\left(k+1\right) - k^2 + 2k(\gamma + \delta + n) + 1\right) \le k^2 - \frac{3k}{2} + \frac{1}{2} < \text{2nk+1}$$

In[•]:=
```
(* Comparison 4th - 4th *)
Print["-2nk-1 < ", TraditionalForm[Simplify[f4[δ1] - f4[δ2]]],
  " < 2nk+1 and ", TraditionalForm[Simplify[f4[δ1] - f4[δ2]] ≠ 0],
  " since ", TraditionalForm[δ1 ≠ δ2]];
```

-2nk-1 < $-(k-1)\ (\delta 1-\delta 2)$ < 2nk+1 and $-(k-1)\ (\delta 1-\delta 2) \neq 0$ since $\delta 1 \neq \delta 2$

In[•]:=
```
(* CASE 2) k even and i even (k odd and i odd) *)

(* We first parametrize each partial sum as follows *)
f1[a_] := -a (k - 1) - k (n - k + 1 + i);
f2[b_] := b (k - 1);
f3[g_] := (k - i) / 2 (k - 1) - g (k - 1) - i;
f4[d_] := (k - i) / 2 (k - 1) + d (k - 1);

Style["Parametric forms for k even and i even", FontWeight → Bold]
Print["1st parametric form: ", TraditionalForm[f1[α]],
  " for ", TraditionalForm[0 ≤ α ≤ (k - i) / 2 - 1]];
Print["2nd parametric form: ", TraditionalForm[f2[β]],
  " for ", TraditionalForm[1 ≤ β ≤ (k - i) / 2]];
Print["3rd parametric form: ", TraditionalForm[f3[γ]],
  " for ", TraditionalForm[0 ≤ γ ≤ i / 2 - 1]];
Print["4th parametric form: ", TraditionalForm[f4[δ]],
  " for ", TraditionalForm[1 ≤ δ ≤ i / 2]];

Style["Parametric forms for k odd and i odd", FontWeight → Bold]
Print["1st parametric form: ", TraditionalForm[-f1[α]],
  " for ", TraditionalForm[0 ≤ α ≤ (k - i) / 2 - 1]];
Print["2nd parametric form: ", TraditionalForm[-f2[β]],
  " for ", TraditionalForm[1 ≤ β ≤ (k - i) / 2]];
Print["3rd parametric form: ", TraditionalForm[-f3[γ]],
  " for ", TraditionalForm[0 ≤ γ ≤ (i - 1) / 2]];
Print["4th parametric form: ", TraditionalForm[-f4[δ]],
  " for ", TraditionalForm[1 ≤ δ ≤ (i - 1) / 2]];
```

Out[•]= **Parametric forms for k even and i even**

1st parametric form: $-k\ (i-k+n+1)-\alpha\ (k-1)$ for $0 \le \alpha \le \frac{k-i}{2}-1$

2nd parametric form: $\beta\ (k-1)$ for $1 \le \beta \le \frac{k-i}{2}$

3rd parametric form: $\frac{1}{2}\ (k-1)\ (k-i)-i-\gamma\ (k-1)$ for $0 \le \gamma \le \frac{i}{2}-1$

4th parametric form: $\frac{1}{2}\ (k-1)\ (k-i)+\delta\ (k-1)$ for $1 \le \delta \le \frac{i}{2}$

Out[•]= **Parametric forms for k odd and i odd**

1st parametric form: $k\left(i-k+n+1\right)+\alpha\left(k-1\right)$ for $0 \le \alpha \le \frac{k-i}{2}-1$

2nd parametric form: $\beta\left(-\left(k-1\right)\right)$ for $1 \le \beta \le \frac{k-i}{2}$

3rd parametric form: $-\frac{1}{2}\left(k-1\right)\left(k-i\right)+i+\gamma\left(k-1\right)$ for $0 \le \gamma \le \frac{i-1}{2}$

4th parametric form: $-\frac{1}{2}\left(k-1\right)\left(k-i\right)-\delta\left(k-1\right)$ for $1 \le \delta \le \frac{i-1}{2}$

In[∘]:=
```
(* Comparison 1st - 1st *)
Print["-2nk-1 < ", TraditionalForm[Simplify[f1[α1] - f1[α2]]],
  " < 2nk+1 and ", TraditionalForm[Simplify[f1[α1] - f1[α2]] ≠ 0],
  " since ", TraditionalForm[α1 ≠ α2]];
```

-2nk-1 < $-\left(k-1\right)\left(\alpha 1-\alpha 2\right)$ < 2nk+1 and $-\left(k-1\right)\left(\alpha 1-\alpha 2\right) \neq 0$ since $\alpha 1 \neq \alpha 2$

In[∘]:=
```
(* Comparison 1st - 2nd *)
max = Maximize[{f2[b] - f1[a], i ≥ 1 , i ≤ k - 1,
    a == (k - i) / 2 - 1, b == (k - i) / 2, k ≥ 1, n ≥ k}, {i, b, a}];
max = max[[1]][[1]][[2]][[1]];
min = Minimize[
   {f2[b] - f1[a], i ≥ 1 , i ≤ k - 1, a == 0, b == 1, k ≥ 1, n ≥ k}, {i, b, a}];
min = min[[1]][[1]][[2]][[1]];
Print["0 < ", TraditionalForm[Simplify[min]],
  " ≤ ", TraditionalForm[Simplify[f2[β] - f1[α]]],
  " ≤ ", TraditionalForm[Simplify[max]], " < 2nk+1"];
```

0 < $-k^2+k\left(n+3\right)-1$ ≤ $-\alpha-\beta+k\left(\alpha+\beta+i+n+1\right)-k^2$ ≤ $k\,n$ < 2nk+1

In[∘]:=
```
(* Comparison 1st - 3rd *)
max = Maximize[{f3[g] - f1[a], i ≥ 1, i ≤ k - 1, a ≥ 0,
    a ≤ (k - i) / 2 - 1, g ≥ 0, g ≤ (i - 1) / 2, k ≥ 1, n ≥ k}, {i, g, a}];
max = max[[1]][[1]][[2]][[1]];
min = Minimize[{f3[g] - f1[a], i ≥ 1, i ≤ k - 1, a ≥ 0,
    a ≤ (k - i) / 2 - 1, g ≥ 0, g ≤ (i - 1) / 2, k ≥ 1, n ≥ k}, {i, g, a}];
min = min[[1]][[1]][[2]][[1]];
Print["0 < ", TraditionalForm[Simplify[min]],
  " ≤ ", TraditionalForm[Simplify[f3[γ] - f1[α]]],
  " ≤ ", TraditionalForm[Simplify[max]], " < 2nk+1"];
```

0 < $-\frac{k^2}{2}+k\,n+k-\frac{1}{2}$ ≤

$\frac{1}{2}\left(-2\,\alpha+2\,\gamma+i\left(k-1\right)-k^2+k\left(2\,\alpha-2\,\gamma+2\,n+1\right)\right)$ ≤ $k\left(n-1\right)+1$ < 2nk+1

In[•]:=
```
(* Comparison 1st - 4th *)
max = Maximize[{f4[d] - f1[a], i ≥ 2 , i ≤ k - 1,
    a == (k - i) / 2 - 1, d == i / 2, k ≥ 1, n ≥ k}, {d, a, i}];
max = max[[1]][[1]][[2]][[1]];
min =
 Minimize[{f4[d] - f1[a], i ≥ 1 , i ≤ k - 1, a == 0, d == 1, k ≥ 1, n ≥ k}, {d, a, i}];
min = min[[1]][[1]][[2]][[1]];
Print["0 < ", TraditionalForm[Simplify[min]],
  " ≤ ", TraditionalForm[Simplify[f4[δ] - f1[α]]],
  " ≤ ", TraditionalForm[Simplify[max]], " < 2nk+1"];
```

$$0 < -\frac{k^2}{2} + k\,(n+2) - \frac{1}{2} \leq$$
$$\frac{1}{2}\left(-2\,(\alpha+\delta) + i\left(k+1\right) - k^2 + k\,(2\,\alpha + 2\,\delta + 2\,n + 1)\right) \leq \frac{1}{2}\left(k^2 + 2\,k\,(n-1) + 1\right) < 2nk+1$$

In[•]:=
```
(* Comparison 2nd - 2nd *)
Print["-2nk-1 < ", TraditionalForm[Simplify[f2[β1] - f2[β2]]],
  " < 2nk+1 and ", TraditionalForm[Simplify[f2[β1] - f2[β2]] ≠ 0],
  " since ", TraditionalForm[β1 ≠ β2]];
```

$$-2nk-1 < \left(k-1\right)(\beta 1 - \beta 2) < 2nk+1 \text{ and } \left(k-1\right)(\beta 1 - \beta 2) \neq 0 \text{ since } \beta 1 \neq \beta 2$$

In[•]:=
```
(* Comparison 2nd - 3rd *)
max = Maximize[{f3[g] - f2[b], i ≥ 1, i ≤ k - 1, b == 1,
    b ≤ (k - i) / 2, g == 0, g ≤ (i - 1) / 2, k ≥ 1, n ≥ k}, {i, g, b}];
max = max[[1]][[1]][[2]][[1]];
min = Minimize[{f3[g] - f2[b], i ≥ 1, i ≤ k - 1,
    b == (k - i) / 2, g == (i - 1) / 2, k ≥ 1, n ≥ k}, {g, b, i}];
min = min[[1]][[1]][[2]][[1]];
Print["-2nk-1 < ", TraditionalForm[Simplify[min]],
  " ≤ ", TraditionalForm[(k - 1) (-β - γ + (k - i)/2) - i],
  " ≤ ", TraditionalForm[Simplify[max]], " < 2nk+1"];
Reduce[{f3[g] - f2[b] == 0 && i ≥ 1 && i == k - 1 && k ≥ 1 && n ≥ k,
   b ≥ 1, b ≤ (k - i) / 2, g ≥ 0, g ≤ (i - 1) / 2}, {n, k, i}, Integers];
Print["Since ", TraditionalForm[(k - 1) (-β - γ + (k - i)/2) - i],
  " = 0 if and only if ", Congruent[i, 0],
  "(mod k-1) then the only possible value for i is k-1 because ",
  TraditionalForm[1 ≤ i ≤ k - 1], "."];
Print["Does ", TraditionalForm[(k - 1) (-β - γ - 1 / 2) ],
  " = 0 have a solution? ", sol, "."];
```

$$-2nk-1 < -\frac{1}{2}\left(k-1\right)k \leq \left(k-1\right)\left(-\beta-\gamma+\frac{k-i}{2}\right) - i \leq \frac{1}{2}\left(k^2 - 4\,k + 1\right) < 2nk+1$$

Since $\left(k-1\right)\left(-\beta-\gamma+\frac{k-i}{2}\right) - i = 0$ if and only if i ≡ 0

(mod k-1) then the only possible value for i is k-1 because $1 \leq i \leq k-1$.

Does $\left(k-1\right)\left(-\beta-\gamma-\frac{1}{2}\right) = 0$ have a solution? False.

In[•]:=
```
(* Comparison 2nd - 4th *)
max = Maximize[{f4[d] - f2[b], i ≥ 1 , i ≤ k - 1,
    b ≥ 1, b ≤ (k - i) / 2, d ≥ 1, d ≤ i / 2, k ≥ 1, n ≥ k}, {d, b, i}];
max = max[[1]][[1]][[2]][[1]];
min = Minimize[{f4[d] - f2[b], i ≥ 1 , i ≤ k - 1,
    b ≥ 1, b ≤ (k - i) / 2, d ≥ 1, d ≤ i / 2, k ≥ 1, n ≥ k}, {d, b, i}];
min = min[[1]][[1]][[2]][[1]];
Print["0 < ", TraditionalForm[Simplify[min]],
  " ≤ ", TraditionalForm[Simplify[f4[δ] - f2[β]]],
  " ≤ ", TraditionalForm[Simplify[max]], " < 2nk+1"];
```

$0 < k-1 \le \frac{1}{2}(k-1)(-2\beta+2\delta-i+k) \le \frac{1}{2}(k^2-3k+2) < \text{2nk+1}$

In[•]:=
```
(* Comparison 3rd - 3rd *)
Print["-2nk-1 < ", TraditionalForm[Simplify[f3[γ1] - f3[γ2]]],
  " < 2nk+1 and ", TraditionalForm[Simplify[f3[γ1] - f3[γ2]] ≠ 0],
  " since ", TraditionalForm[γ1 ≠ γ2]];
```

$\text{-2nk-1} < -(k-1)(\gamma1-\gamma2) < \text{2nk+1}$ and $-(k-1)(\gamma1-\gamma2) \neq 0$ since $\gamma1 \neq \gamma2$

In[•]:=
```
(* Comparison 3rd - 4th *)
max = Maximize[{f4[d] - f3[g], i ≥ 1 ,
    i ≤ k - 1, g == (i - 1) / 2, d == i / 2, k ≥ 1, n ≥ k}, {d, g, i}];
max = max[[1]][[1]][[2]][[1]];
min =
  Minimize[{f4[d] - f3[g], i ≥ 1 , i ≤ k - 1, g == 0, d == 1, k ≥ 1, n ≥ k}, {d, g, i}];
min = min[[1]][[1]][[2]][[1]];
Print["0 < ", TraditionalForm[Simplify[min]],
  " ≤ ", TraditionalForm[Simplify[f4[δ] - f3[γ]]],
  " ≤ ", TraditionalForm[Simplify[max]], " < 2nk+1"];
```

$0 < k \le i+(k-1)(\gamma+\delta) \le k^2-\frac{3k}{2}+\frac{1}{2} < \text{2nk+1}$

In[•]:=
```
(* Comparison 4th - 4th *)
Print["-2nk-1 < ", TraditionalForm[Simplify[f4[δ1] - f4[δ2]]],
  " < 2nk+1 and ", TraditionalForm[Simplify[f4[δ1] - f4[δ2]] ≠ 0],
  " since ", TraditionalForm[δ1 ≠ δ2]];
```

$\text{-2nk-1} < (k-1)(\delta1-\delta2) < \text{2nk+1}$ and $(k-1)(\delta1-\delta2) \neq 0$ since $\delta1 \neq \delta2$