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Title: Mathematica Output for the upper bound when  $B=10$ ,  $K=9$  and  $\epsilon=1/15$

EXACT VALUE OF  $M_1$  (Balanced-Balanced distributions)

Point number 1 of type  $\{\{a,a,a,a,a,a,a,a,a\},\{d,d,d,d,d,d,d,d,d\}\}$  with  $\{a,d\} \leq 1/15$  has maximum value equal to  $\{0.036288,\{a>0.1,d>0.1\}\}$

Point number 2 of type  $\{\{0,a,a,a,a,a,a,a,a\},\{1-9\,d,d,d,d,d,d,d,d,d\}\}$  with  $\{a,d\} \leq 1/15$  has maximum value equal to  $\{0.0359676,\{a>0.111111,d>0.00740741\}\}$

Point number 3 of type  $\{\{0,a,a,a,a,a,a,a,1-8\,a\},\{1-8\,d,d,d,d,d,d,d,d,0\}\}$  with  $\{a,d\} \leq 1/15$  has maximum value equal to  $\{0.0359677,\{a>0.00833333,d>0.111075\}\}$

Point number 4 of type  $\{\{0,0,a,a,a,a,a,a,a\},\{1/2\,(1-8\,d),1/2\,(1-8\,d),d,d,d,d,d,d,d,d\}\}$  with  $\{a,d\} \leq 1/15$  has maximum value equal to  $\{0.0249287,\{a>0.125,d>0.092263\}\}$

Point number 5 of type  $\{\{0,0,a,a,a,a,a,a,1-7\,a\},\{1/2\,(1-7\,d),1/2\,(1-7\,d),d,d,d,d,d,d,d,0\}\}$  with  $\{a,d\} \leq 1/15$  has maximum value equal to  $\{0.0359678,\{a>0.00952381,d>0.111029\}\}$

Point number 6 of type  $\{\{0,0,a,a,a,a,a,a,1/2\,(1-6\,a),1/2\,(1-6\,a)\},\{1/2\,(1-6\,d),1/2\,(1-6\,d),d,d,d,d,d,d,d,0,0\}\}$  with  $\{a,d\} \leq 1/15$  has maximum value equal to  $\{0.018105,\{a>0.0111111,d>0.124938\}\}$

Point number 7 of type  $\{\{0,0,0,a,a,a,a,a,a\},\{1/3\,(1-7\,d),1/3\,(1-7\,d),1/3\,(1-7\,d),d,d,d,d,d,d,d\}\}$  with  $\{a,d\} \leq 1/15$  has maximum value equal to  $\{0.0207378,\{a>0.142857,d>0.0916594\}\}$

Point number 8 of type  $\{\{0,0,0,a,a,a,a,a,1-6\,a\},\{1/3\,(1-6\,d),1/3\,(1-6\,d),1/3\,(1-6\,d),d,d,d,d,d,d,d,0\}\}$  with  $\{a,d\} \leq 1/15$  has maximum value equal to  $\{0.0359679,\{a>0.0111111,d>0.110967\}\}$

Point number 9 of type  $\{\{0,0,0,a,a,a,a,a,1/2\,(1-5\,a),1/2\,(1-5\,a)\},\{1/3\,(1-5\,d),1/3\,(1-5\,d),1/3\,(1-5\,d),d,d,d,d,d,d,d,0,0\}\}$  with  $\{a,d\} \leq 1/15$  has maximum value equal to  $\{0.0181048,\{a>0.0133333,d>0.124889\}\}$

Point number 10 of type  $\{\{0,0,0,a,a,a,a,1/3\,(1-4\,a),1/3\,(1-4\,a),1/3\,(1-4\,a)\},\{1/3\,(1-4\,d),1/3\,(1-4\,d),1/3\,(1-4\,d),d,d,d,d,d,d,d,0,0,0\}\}$  with  $\{a,d\} \leq 1/15$  has maximum value equal to  $\{0.00571692,\{a>0.0166667,d>0.142839\}\}$

Point number 11 of type  $\{\{14/15,a,a,a,a,a,a,a,a\},\{0,d,d,d,d,d,d,d,d\}\}$  with  $\{a,d\} \leq 1/15$  has maximum value equal to  $\{0.0359676,\{a>0.00740741,d>0.111111\}\}$

Point number 12 of type  $\{\{14/15,a,a,a,a,a,a,a,1/15-8\,a\},\{0,d,d,d,d,d,d,d,0\}\}$  with  $\{a,d\} \leq 1/15$  has maximum value equal to  $\{0.0192261,\{a>0.,d>0.125\}\}$

Point number 13 of type  $\{\{14/15,a,a,a,a,a,a,a,1/2\,(1/15-7\,a),1/2\,(1/15-7\,a)\},\{0,d,d,d,d,d,d,d,0,0\}\}$  with  $\{a,d\} \leq 1/15$  has maximum value equal to  $\{0.0061199,\{a>0.,d>0.125\}\}$

$>0.,d->0.142857\}}$

Point number 14 of type  $\{\{14/15,0,a,a,a,a,a,a,a\},\{0,1-8\,d,d,d,d,d,d,d,d\}\}$  with  $\{a,d\} \leq 1/15$  has maximum value equal to  $\{0.0359677,\{a->0.00833333,d->0.111075\}\}$

Point number 15 of type  $\{\{14/15,0,a,a,a,a,a,a,1/15-7\,a\},\{0,1-7\,d,d,d,d,d,d,d,0\}\}$  with  $\{a,d\} \leq 1/15$  has maximum value equal to  $\{0.0192261,\{a->0.,d->0.125\}\}$

Point number 16 of type  $\{\{14/15,0,a,a,a,a,a,a,1/2\,(1/15-6\,a),1/2\,(1/15-6\,a)\},\{0,1-6\,d,d,d,d,d,d,d,0,0\}\}$  with  $\{a,d\} \leq 1/15$  has maximum value equal to  $\{0.0061199,\{a->0.,d->0.142857\}\}$

Point number 17 of type  $\{\{14/15,0,0,a,a,a,a,a,a\},\{0,1/2\,(1-7\,d),1/2\,(1-7\,d),d,d,d,d,d,d,d\}\}$  with  $\{a,d\} \leq 1/15$  has maximum value equal to  $\{0.0359678,\{a->0.00952381,d->0.111029\}\}$

Point number 18 of type  $\{\{14/15,0,0,a,a,a,a,a,1/15-6\,a\},\{0,1/2\,(1-6\,d),1/2\,(1-6\,d),d,d,d,d,d,d,0\}\}$  with  $\{a,d\} \leq 1/15$  has maximum value equal to  $\{0.0192261,\{a->0.,d->0.125\}\}$

Point number 19 of type  $\{\{14/15,0,0,a,a,a,a,a,1/2\,(1/15-5\,a),1/2\,(1/15-5\,a)\},\{0,1/2\,(1-5\,d),1/2\,(1-5\,d),d,d,d,d,d,d,0\}\}$  with  $\{a,d\} \leq 1/15$  has maximum value equal to  $\{0.0061199,\{a->0.,d->0.142857\}\}$

Point number 20 of type  $\{\{14/15,0,0,0,a,a,a,a,a\},\{0,1/3\,(1-6\,d),1/3\,(1-6\,d),1/3\,(1-6\,d),d,d,d,d,d,d\}\}$  with  $\{a,d\} \leq 1/15$  has maximum value equal to  $\{0.0359679,\{a->0.0111111,d->0.110967\}\}$

Point number 21 of type  $\{\{14/15,0,0,0,a,a,a,a,a,1/15-5\,a\},\{0,1/3\,(1-5\,d),1/3\,(1-5\,d),1/3\,(1-5\,d),d,d,d,d,d,d,0\}\}$  with  $\{a,d\} \leq 1/15$  has maximum value equal to  $\{0.0192261,\{a->0.,d->0.125\}\}$

Point number 22 of type  $\{\{14/15,0,0,0,a,a,a,a,1/2\,(1/15-4\,a),1/2\,(1/15-4\,a)\},\{0,1/3\,(1-4\,d),1/3\,(1-4\,d),1/3\,(1-4\,d),d,d,d,d,d,d,0\}\}$  with  $\{a,d\} \leq 1/15$  has maximum value equal to  $\{0.0061199,\{a->0.,d->0.142857\}\}$

Point number 23 of type  $\{\{14/15,a,a,a,a,a,a,a,0\},\{0,d,d,d,d,d,d,d,14/15\}\}$  with  $\{a,d\} \leq 1/15$  has maximum value equal to  $\{8.40211*10^{-8},\{a->0.00833333,d->0.00833333\}\}$

Point number 24 of type  $\{\{14/15,0,a,a,a,a,a,a,0\},\{0,1/15-7\,d,d,d,d,d,d,d,14/15\}\}$  with  $\{a,d\} \leq 1/15$  has maximum value equal to  $\{6.53885*10^{-8},\{a->0.00952381,d->0.00832102\}\}$

Point number 25 of type  $\{\{14/15,0,a,a,a,a,a,1/15-6\,a,0\},\{0,1/15-6\,d,d,d,d,d,d,d,14/15\}\}$  with  $\{a,d\} \leq 1/15$  has maximum value equal to  $\{4.68061*10^{-8},\{a->0.00950466,d->0.00950466\}\}$

Point number 26 of type  $\{\{14/15,0,0,a,a,a,a,a,0\},\{0,1/2\,(1/15-6\,d),1/2\,(1/15-6\,d),d,d,d,d,d,d,14/15\}\}$  with  $\{a,d\} \leq 1/15$  has maximum value equal to

{5.04204\*10<sup>-8</sup>, {a->0.0111111, d->0.00831681}}

Point number 27 of type {{14/15, 0, 0, a, a, a, a, 1/15-5 a, 0}, {0, 1/2 (1/15-5 d), 1/2 (1/15-5 d), d, d, d, d, 0, 14/15}} with {a, d} <= 1/15 has maximum value equal to {3.18684\*10<sup>-8</sup>, {a->0.0110587, d->0.00950381}}

Point number 28 of type {{14/15, 0, 0, a, a, a, a, 1/2 (1/15-4 a), 1/2 (1/15-4 a), 0}, {0, 1/2 (1/15-4 d), 1/2 (1/15-4 d), d, d, d, d, 0, 14/15}} with {a, d} <= 1/15 has maximum value equal to {1.69175\*10<sup>-8</sup>, {a->0.0110679, d->0.0110679}}

Point number 29 of type {{14/15, 1/15, 0, 0, 0, 0, 0, 0, 0}, {0, a, (1-a)/8, (1-a)/8, (1-a)/8, (1-a)/8, (1-a)/8, (1-a)/8, (1-a)/8}} with {a} <= 1/15 has maximum value equal to {0.0359729, {a->0.108753}}

Point number 30 of type {{14/15, 1/15, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 1/8, 1/8, 1/8, 1/8, 1/8, 1/8, 1/8}} with {} <= 1/15 has maximum value equal to 0.01922607421875`

Point number 31 of type {{14/15, 1/15, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 1/7, 1/7, 1/7, 1/7, 1/7, 1/7}} with {} <= 1/15 has maximum value equal to 0.006119899021666143`

Max attained in point 1 with value 0.036288`.

EXACT VALUE OF M\_2 (Unbalanced-Balanced distributions)

Point number 1 of type {{a, a, a, a, a, a, a, a, a}, {d, d, d, d, d, d, d, d, d}} with value {0.036288, {a->0.1, d->0.1}}

Point number 2 of type {{0, a, a, a, a, a, a, a, a}, {1-9 d, d, d, d, d, d, d, d, d}} with value {0.0379346, {a->0.111111, d->0.}}

Point number 3 of type {{0, a, a, a, a, a, a, a, 1-8 a}, {1-8 d, d, d, d, d, d, d, d, 0}} with value {0.0379346, {a->0., d->0.111111}}

Point number 4 of type {{0, 0, a, a, a, a, a, a, a}, {1/2 (1-8 d), 1/2 (1-8 d), d, d, d, d, d, d, d}} with value {0.0249287, {a->0.125, d->0.092263}}

Point number 5 of type {{0, 0, a, a, a, a, a, a, 1-7 a}, {1/2 (1-7 d), 1/2 (1-7 d), d, d, d, d, d, d, 0}} with value {0.0379346, {a->0., d->0.111111}}

Point number 6 of type {{0, 0, a, a, a, a, a, 1/2 (1-6 a), 1/2 (1-6 a)}, {1/2 (1-6 d), 1/2 (1-6 d), d, d, d, d, d, 0, 0}} with value {0.0192261, {a->0., d->0.125}}

Point number 7 of type {{0, 0, 0, a, a, a, a, a, a}, {1/3 (1-7 d), 1/3 (1-7 d), 1/3 (1-7 d), d, d, d, d, d, d, d}} with value {0.0207378, {a->0.142857, d->0.0916594}}

Point number 8 of type {{0, 0, 0, a, a, a, a, a, 1-6 a}, {1/3 (1-6 d), 1/3 (1-6 d), 1/3 (1-6 d), d, d, d, d, d, 0}} with value {0.0379346, {a->0., d->0.111111}}

Point number 9 of type  $\{0,0,0,a,a,a,a,1/2(1-5a),1/2(1-5a)\},\{1/3(1-5d),1/3(1-5d),1/3(1-5d),d,d,d,d,0,0\}$  with value  $\{0.0192261,\{a>0.,d>0.125\}\}$

Point number 10 of type  $\{0,0,0,a,a,a,a,1/3(1-4a),1/3(1-4a),1/3(1-4a)\},\{1/3(1-4d),1/3(1-4d),1/3(1-4d),d,d,d,d,0,0\}$  with value  $\{0.0061199,\{a>0.,d>0.142857\}\}$

Max attained in point 2 with value 0.037934596690883844`.

EXACT VALUE OF  $M_3$  (Unbalanced-Unbalanced distributions on the same coordinate)

Point number 1 of type  $\{14/15,a,a,a,a,a,a,a,a\},\{14/15,d,d,d,d,d,d,d,d\}$  has maximum value equal to  $\{3.32211*10^{-9},\{a>0.00740741,d>0.00740741\}\}$

Point number 2 of type  $\{14/15,0,a,a,a,a,a,a,a\},\{14/15,1/15-8d,d,d,d,d,d,d,d\}$  has maximum value equal to  $\{3.0532*10^{-9},\{a>0.00833333,d>0.\}\}$

Point number 3 of type  $\{14/15,0,a,a,a,a,a,a,1/15-7a\},\{14/15,1/15-7d,d,d,d,d,d,d,d\}$  has maximum value equal to  $\{3.0532*10^{-9},\{a>0.,d>0.00833333\}\}$

Point number 4 of type  $\{14/15,0,0,a,a,a,a,a,a\},\{14/15,1/2(1/15-7d),1/2(1/15-7d),d,d,d,d,d,d,d\}$  has maximum value equal to  $\{2.33635*10^{-9},\{a>0.00952381,d>0.00673184\}\}$

Point number 5 of type  $\{14/15,0,0,a,a,a,a,a,1/15-6a\},\{14/15,1/2(1/15-6d),1/2(1/15-6d),d,d,d,d,d,d,d\}$  has maximum value equal to  $\{3.0532*10^{-9},\{a>0.,d>0.00833333\}\}$

Point number 6 of type  $\{14/15,0,0,a,a,a,a,a,1/2(1/15-5a),1/2(1/15-5a)\},\{14/15,1/2(1/15-5d),1/2(1/15-5d),d,d,d,d,d,d,d\}$  has maximum value equal to  $\{1.67391*10^{-9},\{a>0.,d>0.00952381\}\}$

Point number 7 of type  $\{14/15,0,0,0,a,a,a,a,a\},\{14/15,1/3(1/15-6d),1/3(1/15-6d),1/3(1/15-6d),d,d,d,d,d,d\}$  has maximum value equal to  $\{1.93337*10^{-9},\{a>0.0111111,d>0.00664556\}\}$

Point number 8 of type  $\{14/15,0,0,0,a,a,a,a,1/15-5a\},\{14/15,1/3(1/15-5d),1/3(1/15-5d),1/3(1/15-5d),d,d,d,d,d,d\}$  has maximum value equal to  $\{3.0532*10^{-9},\{a>0.,d>0.00833333\}\}$

Point number 9 of type  $\{14/15,0,0,0,a,a,a,a,1/2(1/15-4a),1/2(1/15-4a)\},\{14/15,1/3(1/15-4d),1/3(1/15-4d),1/3(1/15-4d),d,d,d,d,d,d\}$  has maximum value equal to  $\{1.67391*10^{-9},\{a>0.,d>0.00952381\}\}$

Point number 10 of type  $\{14/15,0,0,0,a,a,a,1/3(1/15-3a),1/3(1/15-3a),1/3(1/15-3a)\},\{14/15,1/3(1/15-3d),1/3(1/15-3d),1/3(1/15-3d),d,d,d,d,d,d\}$  has maximum value equal to  $\{6.3394*10^{-10},\{a>0.00952381,d>0.00952381\}\}$

Point number 11 of type  $\{14/15,1/15,0,0,0,0,0,0,0\},\{14/15,a,1/8(1/15-a),1/8(1/15-$

a), 1/8 (1/15-a), 1/8 (1/15-a), 1/8 (1/15-a), 1/8 (1/15-a), 1/8 (1/15-a), 1/8 (1/15-a)} has maximum value equal to  $\{3.0532 \cdot 10^{-9}, \{a > 0\}\}$

Point number 12 of type  $\{\{14/15, 1/15, 0, 0, 0, 0, 0, 0, 0\}, \{14/15, 0, 1/120, 1/120, 1/120, 1/120, 1/120, 1/120, 1/120, 1/120\}\}$  has maximum value equal to  $3.0531978737997257 \cdot 10^{-9}$

Point number 13 of type  $\{\{14/15, 1/15, 0, 0, 0, 0, 0, 0, 0\}, \{14/15, 0, 0, 1/105, 1/105, 1/105, 1/105, 1/105, 1/105, 1/105\}\}$  has maximum value equal to  $1.6739103777352832 \cdot 10^{-9}$

Point number 14 of type  $\{\{14/15, 1/15, 0, 0, 0, 0, 0, 0, 0\}, \{14/15, 0, 0, 0, 1/90, 1/90, 1/90, 1/90, 1/90, 1/90\}\}$  has maximum value equal to  $5.900937263026375 \cdot 10^{-10}$

Max attained in point 1 with value  $3.3221119139680353 \cdot 10^{-9}$ .

EXACT VALUE OF M\_4 (Unbalanced-Unbalanced distributions on different coordinates)

Point number 1 of type  $\{\{14/15, a, a, a, a, a, a, a, 0\}, \{0, d, d, d, d, d, d, d, 14/15\}\}$  has maximum value equal to  $\{8.40211 \cdot 10^{-8}, \{a > 0.00833333, d > 0.00833333\}\}$

Point number 2 of type  $\{\{14/15, 0, a, a, a, a, a, a, 0\}, \{0, 1/15-7 d, d, d, d, d, d, d, 14/15\}\}$  has maximum value equal to  $\{6.53885 \cdot 10^{-8}, \{a > 0.00952381, d > 0.00832102\}\}$

Point number 3 of type  $\{\{14/15, 0, a, a, a, a, a, 1/15-6 a, 0\}, \{0, 1/15-6 d, d, d, d, d, d, 0, 14/15\}\}$  has maximum value equal to  $\{4.68061 \cdot 10^{-8}, \{a > 0.00950466, d > 0.00950466\}\}$

Point number 4 of type  $\{\{14/15, 0, 0, a, a, a, a, a, 0\}, \{0, 1/2 (1/15-6 d), 1/2 (1/15-6 d), d, d, d, d, d, 14/15\}\}$  has maximum value equal to  $\{5.04204 \cdot 10^{-8}, \{a > 0.0111111, d > 0.00831681\}\}$

Point number 5 of type  $\{\{14/15, 0, 0, a, a, a, a, 1/15-5 a, 0\}, \{0, 1/2 (1/15-5 d), 1/2 (1/15-5 d), d, d, d, d, d, 0, 14/15\}\}$  has maximum value equal to  $\{3.18684 \cdot 10^{-8}, \{a > 0.0110587, d > 0.00950381\}\}$

Point number 6 of type  $\{\{14/15, 0, 0, a, a, a, 1/2 (1/15-4 a), 1/2 (1/15-4 a), 0\}, \{0, 1/2 (1/15-4 d), 1/2 (1/15-4 d), d, d, d, d, 0, 14/15\}\}$  has maximum value equal to  $\{1.69175 \cdot 10^{-8}, \{a > 0.0110679, d > 0.0110679\}\}$

Point number 7 of type  $\{\{1, 0, 0, 0, 0, 0, 0, 0, 0\}, \{0, 1/120, 1/120, 1/120, 1/120, 1/120, 1/120, 1/120, 1/120, 14/15\}\}$  has maximum value equal to  $4.422260802469136 \cdot 10^{-8}$

Point number 8 of type  $\{\{1, 0, 0, 0, 0, 0, 0, 0, 0\}, \{0, 0, 1/105, 1/105, 1/105, 1/105, 1/105, 1/105, 1/105, 14/15\}\}$  has maximum value equal to  $2.4607198919489433 \cdot 10^{-8}$

Point number 9 of type  $\{1,0,0,0,0,0,0,0,0\}$ ,  
 $\{0,0,0,1/90,1/90,1/90,1/90,1/90,1/90,14/15\}$  has maximum value equal to  
 $8.851405894539564 \cdot 10^{-9}$

Max attained in point 1 with value  $8.402107981824417 \cdot 10^{-8}$ .

#### UPPER BOUND ON THE RATE

The bound on the rate for  $b=10$ ,  $k=9$  and  $\epsilon=1/15$  is  $0.027727818444739292$   
and  $\eta_0=0.9583994399230845$