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Title: Mathematica Output for the upper bound when $B=9$, $K=8$ and $\epsilon=1/10$

EXACT VALUE OF M_1 (Balanced-Balanced distributions)

Point number 1 of type $\{\{a,a,a,a,a,a,a,a\},\{d,d,d,d,d,d,d,d\}\}$ with $\{a,d\} \leq 1/10$ has maximum value equal to $\{0.0758692,\{a>0.111111,d>0.111111\}\}$

Point number 2 of type $\{\{0,a,a,a,a,a,a,a\},\{1-8\,d,d,d,d,d,d,d,d\}\}$ with $\{a,d\} \leq 1/10$ has maximum value equal to $\{0.0711406,\{a>0.125,d>0.0125\}\}$

Point number 3 of type $\{\{0,a,a,a,a,a,a,1-7\,a\},\{1-7\,d,d,d,d,d,d,d,0\}\}$ with $\{a,d\} \leq 1/10$ has maximum value equal to $\{0.0711402,\{a>0.0142857,d>0.124904\}\}$

Point number 4 of type $\{\{0,0,a,a,a,a,a,a\},\{1/2\,(1-7\,d),1/2\,(1-7\,d),d,d,d,d,d,d\}\}$ with $\{a,d\} \leq 1/10$ has maximum value equal to $\{0.0551281,\{a>0.142857,d>0.0991769\}\}$

Point number 5 of type $\{\{0,0,a,a,a,a,a,1-6\,a\},\{1/2\,(1-6\,d),1/2\,(1-6\,d),d,d,d,d,d,d,0\}\}$ with $\{a,d\} \leq 1/10$ has maximum value equal to $\{0.0711398,\{a>0.0166667,d>0.124775\}\}$

Point number 6 of type $\{\{0,0,a,a,a,a,1/2\,(1-5\,a),1/2\,(1-5\,a)\},\{1/2\,(1-5\,d),1/2\,(1-5\,d),d,d,d,d,d,d,0,0\}\}$ with $\{a,d\} \leq 1/10$ has maximum value equal to $\{0.0392184,\{a>0.02,d>0.142659\}\}$

Point number 7 of type $\{\{0,0,0,a,a,a,a,a\},\{1/3\,(1-6\,d),1/3\,(1-6\,d),1/3\,(1-6\,d),d,d,d,d,d,d\}\}$ with $\{a,d\} \leq 1/10$ has maximum value equal to $\{0.0453803,\{a>0.166667,d>0.0978124\}\}$

Point number 8 of type $\{\{0,0,0,a,a,a,a,1-5\,a\},\{1/3\,(1-5\,d),1/3\,(1-5\,d),1/3\,(1-5\,d),d,d,d,d,d,d,0\}\}$ with $\{a,d\} \leq 1/10$ has maximum value equal to $\{0.0711399,\{a>0.02,d>0.124595\}\}$

Point number 9 of type $\{\{0,0,0,a,a,a,1/2\,(1-4\,a),1/2\,(1-4\,a)\},\{1/3\,(1-4\,d),1/3\,(1-4\,d),1/3\,(1-4\,d),d,d,d,d,d,d,0,0\}\}$ with $\{a,d\} \leq 1/10$ has maximum value equal to $\{0.0391926,\{a>0.025,d>0.142505\}\}$

Point number 10 of type $\{\{0,0,0,a,a,a,1/3\,(1-3\,a),1/3\,(1-3\,a),1/3\,(1-3\,a)\},\{1/3\,(1-3\,d),1/3\,(1-3\,d),1/3\,(1-3\,d),d,d,d,d,d,d,0,0,0\}\}$ with $\{a,d\} \leq 1/10$ has maximum value equal to $\{0.0165788,\{a>0.142857,d>0.142857\}\}$

Point number 11 of type $\{\{9/10,a,a,a,a,a,a,a\},\{0,d,d,d,d,d,d,d\}\}$ with $\{a,d\} \leq 1/10$ has maximum value equal to $\{0.0711406,\{a>0.0125,d>0.125\}\}$

Point number 12 of type $\{\{9/10,a,a,a,a,a,a,1/10-7\,a\},\{0,d,d,d,d,d,d,d,0\}\}$ with $\{a,d\} \leq 1/10$ has maximum value equal to $\{0.0428393,\{a>0.,d>0.142857\}\}$

Point number 13 of type $\{\{9/10,a,a,a,a,a,a,1/2\,(1/10-6\,a),1/2\,(1/10-6\,a)\},\{0,d,d,d,d,d,d,d,0,0\}\}$ with $\{a,d\} \leq 1/10$ has maximum value equal to $\{0.0154321,\{a>$

$\{a > 0, d > 0.166667\}$

Point number 14 of type $\{9/10, 0, a, a, a, a, a, a\}, \{0, 1-7 d, d, d, d, d, d, d\}$ with $\{a, d\} \leq 1/10$ has maximum value equal to $\{0.0711402, \{a > 0.0142857, d > 0.124904\}\}$

Point number 15 of type $\{9/10, 0, a, a, a, a, a, 1/10-6 a\}, \{0, 1-6 d, d, d, d, d, d, 0\}$ with $\{a, d\} \leq 1/10$ has maximum value equal to $\{0.0428393, \{a > 0, d > 0.142857\}\}$

Point number 16 of type $\{9/10, 0, a, a, a, a, a, 1/2 (1/10-5 a)\}, \{0, 1-5 d, d, d, d, d, d, 0\}$ with $\{a, d\} \leq 1/10$ has maximum value equal to $\{0.0154321, \{a > 0, d > 0.166667\}\}$

Point number 17 of type $\{9/10, 0, 0, a, a, a, a, a\}, \{0, 1/2 (1-6 d), 1/2 (1-6 d), d, d, d, d, d\}$ with $\{a, d\} \leq 1/10$ has maximum value equal to $\{0.0711398, \{a > 0.0166667, d > 0.124775\}\}$

Point number 18 of type $\{9/10, 0, 0, a, a, a, a, 1/10-5 a\}, \{0, 1/2 (1-5 d), 1/2 (1-5 d), d, d, d, d, d, 0\}$ with $\{a, d\} \leq 1/10$ has maximum value equal to $\{0.0428393, \{a > 0, d > 0.142857\}\}$

Point number 19 of type $\{9/10, 0, 0, a, a, a, a, 1/2 (1/10-4 a)\}, \{0, 1/2 (1-4 d), 1/2 (1-4 d), d, d, d, d, 0\}$ with $\{a, d\} \leq 1/10$ has maximum value equal to $\{0.0154321, \{a > 0, d > 0.166667\}\}$

Point number 20 of type $\{9/10, 0, 0, 0, a, a, a, a\}, \{0, 1/3 (1-5 d), 1/3 (1-5 d), 1/3 (1-5 d), d, d, d, d, d\}$ with $\{a, d\} \leq 1/10$ has maximum value equal to $\{0.0711399, \{a > 0.02, d > 0.124595\}\}$

Point number 21 of type $\{9/10, 0, 0, 0, a, a, a, 1/10-4 a\}, \{0, 1/3 (1-4 d), 1/3 (1-4 d), 1/3 (1-4 d), d, d, d, d, 0\}$ with $\{a, d\} \leq 1/10$ has maximum value equal to $\{0.0428393, \{a > 0, d > 0.142857\}\}$

Point number 22 of type $\{9/10, 0, 0, 0, a, a, a, 1/2 (1/10-3 a)\}, \{0, 1/3 (1-3 d), 1/3 (1-3 d), 1/3 (1-3 d), d, d, d, 0, 0\}$ with $\{a, d\} \leq 1/10$ has maximum value equal to $\{0.0154321, \{a > 0, d > 0.166667\}\}$

Point number 23 of type $\{9/10, a, a, a, a, a, a, 0\}, \{0, d, d, d, d, d, d, 9/10\}$ with $\{a, d\} \leq 1/10$ has maximum value equal to $\{0.0000151149, \{a > 0.0142857, d > 0.0142857\}\}$

Point number 24 of type $\{9/10, 0, a, a, a, a, a, 0\}, \{0, 1/10-6 d, d, d, d, d, d, 9/10\}$ with $\{a, d\} \leq 1/10$ has maximum value equal to $\{0.0000122154, \{a > 0.0166667, d > 0.0142337\}\}$

Point number 25 of type $\{9/10, 0, a, a, a, a, a, 1/10-5 a\}, \{0, 1/10-5 d, d, d, d, d, 0, 9/10\}$ with $\{a, d\} \leq 1/10$ has maximum value equal to $\{9.33572 \cdot 10^{-6}, \{a > 0.0165917, d > 0.0165917\}\}$

Point number 26 of type $\{9/10, 0, 0, a, a, a, a, 0\}, \{0, 1/2 (1/10-5 d), 1/2 (1/10-5 d), d, d, d, d, d, 9/10\}$ with $\{a, d\} \leq 1/10$ has maximum value equal to $\{9.48387 \cdot 10^{-6}, \{a > 0.02, d > 0.0142086\}\}$

Point number 27 of type $\{\{9/10, 0, 0, a, a, a, 1/10-4 a, 0\}, \{0, 1/2 (1/10-4 d), 1/2 (1/10-4 d), d, d, d, d, 0, 9/10\}\}$ with $\{a, d\} \leq 1/10$ has maximum value equal to $\{6.61825 \cdot 10^{-6}, \{a > 0.0198227, d > 0.0165759\}\}$

Point number 28 of type $\{\{9/10, 0, 0, a, a, a, 1/2 (1/10-3 a), 1/2 (1/10-3 a), 0\}, \{0, 1/2 (1/10-3 d), 1/2 (1/10-3 d), d, d, d, 0, 9/10\}\}$ with $\{a, d\} \leq 1/10$ has maximum value equal to $\{3.89943 \cdot 10^{-6}, \{a > 0.0198292, d > 0.0198292\}\}$

Point number 29 of type $\{\{9/10, 1/10, 0, 0, 0, 0, 0, 0\}, \{0, a, (1-a)/7, (1-a)/7, (1-a)/7, (1-a)/7, (1-a)/7, (1-a)/7\}\}$ with $\{a\} \leq 1/10$ has maximum value equal to $\{0.0711689, \{a > 0.120112\}\}$

Point number 30 of type $\{\{9/10, 1/10, 0, 0, 0, 0, 0, 0\}, \{0, 0, 1/7, 1/7, 1/7, 1/7, 1/7, 1/7\}\}$ with $\{ \} \leq 1/10$ has maximum value equal to 0.042839293151662995

Point number 31 of type $\{\{9/10, 1/10, 0, 0, 0, 0, 0, 0\}, \{0, 0, 0, 1/6, 1/6, 1/6, 1/6, 1/6\}\}$ with $\{ \} \leq 1/10$ has maximum value equal to 0.015432098765432098

Max attained in point 1 with value 0.07586919338176769 .

EXACT VALUE OF M_2 (Unbalanced-Balanced distributions)

Point number 1 of type $\{\{a, a, a, a, a, a, a, a\}, \{d, d, d, d, d, d, d, d\}\}$ with value $\{0.0758692, \{a > 0.111111, d > 0.111111\}\}$

Point number 2 of type $\{\{0, a, a, a, a, a, a, a\}, \{1-8 d, d, d, d, d, d, d, d\}\}$ with value $\{0.0769043, \{a > 0.125, d > 0.\}\}$

Point number 3 of type $\{\{0, a, a, a, a, a, a, 1-7 a\}, \{1-7 d, d, d, d, d, d, d, 0\}\}$ with value $\{0.0769043, \{a > 0., d > 0.125\}\}$

Point number 4 of type $\{\{0, 0, a, a, a, a, a, a\}, \{1/2 (1-7 d), 1/2 (1-7 d), d, d, d, d, d, d\}\}$ with value $\{0.0551281, \{a > 0.142857, d > 0.0991769\}\}$

Point number 5 of type $\{\{0, 0, a, a, a, a, a, 1-6 a\}, \{1/2 (1-6 d), 1/2 (1-6 d), d, d, d, d, d, 0\}\}$ with value $\{0.0769043, \{a > 0., d > 0.125\}\}$

Point number 6 of type $\{\{0, 0, a, a, a, a, 1/2 (1-5 a), 1/2 (1-5 a)\}, \{1/2 (1-5 d), 1/2 (1-5 d), d, d, d, d, d, 0\}\}$ with value $\{0.0428393, \{a > 0., d > 0.142857\}\}$

Point number 7 of type $\{\{0, 0, 0, a, a, a, a, a\}, \{1/3 (1-6 d), 1/3 (1-6 d), 1/3 (1-6 d), d, d, d, d, d, d\}\}$ with value $\{0.0453803, \{a > 0.166667, d > 0.0978124\}\}$

Point number 8 of type $\{\{0, 0, 0, a, a, a, a, 1-5 a\}, \{1/3 (1-5 d), 1/3 (1-5 d), 1/3 (1-5 d), d, d, d, d, d, 0\}\}$ with value $\{0.0769043, \{a > 0., d > 0.125\}\}$

Point number 9 of type $\{\{0, 0, 0, a, a, a, 1/2 (1-4 a), 1/2 (1-4 a)\}, \{1/3 (1-4 d), 1/3 (1-4 d), 1/3 (1-4 d), d, d, d, d, 0, 0\}\}$ with value $\{0.0428393, \{a > 0., d > 0.142857\}\}$

Point number 10 of type $\{ \{0,0,0,a,a,a,1/3(1-3a),1/3(1-3a),1/3(1-3a)\}, \{1/3(1-3d),1/3(1-3d),1/3(1-3d),d,d,d,0,0,0\} \}$ with value $\{0.0165788, \{a \rightarrow 0.142857, d \rightarrow 0.142857\} \}$

Max attained in point 2 with value 0.076904296875`.

EXACT VALUE OF M_3 (Unbalanced-Unbalanced distributions on the same coordinate)

Point number 1 of type $\{ \{9/10,a,a,a,a,a,a,a\}, \{9/10,d,d,d,d,d,d,d\} \}$ has maximum value equal to $\{9.72839 \cdot 10^{-7}, \{a \rightarrow 0.0125, d \rightarrow 0.0125\} \}$

Point number 2 of type $\{ \{9/10,0,a,a,a,a,a,a\}, \{9/10,1/10-7d,d,d,d,d,d,d\} \}$ has maximum value equal to $\{8.59429 \cdot 10^{-7}, \{a \rightarrow 0.0142857, d \rightarrow 0.0114851\} \}$

Point number 3 of type $\{ \{9/10,0,a,a,a,a,a,1/10-6a\}, \{9/10,1/10-6d,d,d,d,d,d,0\} \}$ has maximum value equal to $\{8.52502 \cdot 10^{-7}, \{a \rightarrow 0., d \rightarrow 0.0142857\} \}$

Point number 4 of type $\{ \{9/10,0,0,a,a,a,a,a\}, \{9/10,1/2(1/10-6d),1/2(1/10-6d),d,d,d,d,d,d\} \}$ has maximum value equal to $\{7.26052 \cdot 10^{-7}, \{a \rightarrow 0.0166667, d \rightarrow 0.0108847\} \}$

Point number 5 of type $\{ \{9/10,0,0,a,a,a,a,1/10-5a\}, \{9/10,1/2(1/10-5d),1/2(1/10-5d),d,d,d,d,d,0\} \}$ has maximum value equal to $\{8.52502 \cdot 10^{-7}, \{a \rightarrow 0., d \rightarrow 0.0142857\} \}$

Point number 6 of type $\{ \{9/10,0,0,a,a,a,1/2(1/10-4a),1/2(1/10-4a)\}, \{9/10,1/2(1/10-4d),1/2(1/10-4d),d,d,d,d,0,0\} \}$ has maximum value equal to $\{5.21678 \cdot 10^{-7}, \{a \rightarrow 0.0142857, d \rightarrow 0.0142857\} \}$

Point number 7 of type $\{ \{9/10,0,0,0,a,a,a,a\}, \{9/10,1/3(1/10-5d),1/3(1/10-5d),1/3(1/10-5d),d,d,d,d,d\} \}$ has maximum value equal to $\{5.97074 \cdot 10^{-7}, \{a \rightarrow 0.02, d \rightarrow 0.0105735\} \}$

Point number 8 of type $\{ \{9/10,0,0,0,a,a,a,1/10-4a\}, \{9/10,1/3(1/10-4d),1/3(1/10-4d),1/3(1/10-4d),d,d,d,d,0\} \}$ has maximum value equal to $\{8.52502 \cdot 10^{-7}, \{a \rightarrow 0., d \rightarrow 0.0142857\} \}$

Point number 9 of type $\{ \{9/10,0,0,0,a,a,a,1/2(1/10-3a),1/2(1/10-3a)\}, \{9/10,1/3(1/10-3d),1/3(1/10-3d),1/3(1/10-3d),d,d,d,0,0\} \}$ has maximum value equal to $\{5.15432 \cdot 10^{-7}, \{a \rightarrow 0., d \rightarrow 0.0166667\} \}$

Point number 10 of type $\{ \{9/10,0,0,0,a,a,1/3(1/10-2a),1/3(1/10-2a),1/3(1/10-2a)\}, \{9/10,1/3(1/10-2d),1/3(1/10-2d),1/3(1/10-2d),d,d,0,0,0\} \}$ has maximum value equal to $\{2.63374 \cdot 10^{-7}, \{a \rightarrow 0.0166667, d \rightarrow 0.0166667\} \}$

Point number 11 of type $\{ \{9/10,1/10,0,0,0,0,0,0,0\}, \{9/10,a,1/7(1/10-a),1/7(1/10-a),1/7(1/10-a),1/7(1/10-a),1/7(1/10-a)\} \}$ has maximum value equal to $\{8.52502 \cdot 10^{-7}, \{a \rightarrow 0.\} \}$

Point number 12 of type $\{ \{9/10,1/10,0,0,0,0,0,0,0\},$

{9/10,0,1/70,1/70,1/70,1/70,1/70,1/70,1/70}} has maximum value equal to $8.525019337180937 \times 10^{-7}$

Point number 13 of type {{9/10,1/10,0,0,0,0,0,0,0},
{9/10,0,0,1/60,1/60,1/60,1/60,1/60,1/60}} has maximum value equal to $5.154320987654321 \times 10^{-7}$

Point number 14 of type {{9/10,1/10,0,0,0,0,0,0,0},
{9/10,0,0,0,1/50,1/50,1/50,1/50,1/50}} has maximum value equal to 2.0736×10^{-7}

Max attained in point 1 with value $9.7283935546875 \times 10^{-7}$.

EXACT VALUE OF M_4 (Unbalanced-Unbalanced distributions on different coordinates)

Point number 1 of type {{9/10,a,a,a,a,a,a,0},{0,d,d,d,d,d,d,9/10}} has maximum value equal to {0.0000151149,{a->0.0142857,d->0.0142857}}

Point number 2 of type {{9/10,0,a,a,a,a,a,0},{0,1/10-6 d,d,d,d,d,d,9/10}} has maximum value equal to {0.0000122154,{a->0.0166667,d->0.0142337}}

Point number 3 of type {{9/10,0,a,a,a,a,a,1/10-5 a,0},{0,1/10-5 d,d,d,d,d,d,0,9/10}} has maximum value equal to {9.33572*10⁻⁶,{a->0.0165917,d->0.0165917}}

Point number 4 of type {{9/10,0,0,a,a,a,a,0},{0,1/2 (1/10-5 d),1/2 (1/10-5 d),d,d,d,d,d,9/10}} has maximum value equal to {9.48387*10⁻⁶,{a->0.02,d->0.0142086}}

Point number 5 of type {{9/10,0,0,a,a,a,a,1/10-4 a,0},{0,1/2 (1/10-4 d),1/2 (1/10-4 d),d,d,d,d,0,9/10}} has maximum value equal to {6.61825*10⁻⁶,{a->0.0198227,d->0.0165759}}

Point number 6 of type {{9/10,0,0,a,a,a,1/2 (1/10-3 a),1/2 (1/10-3 a),0},{0,1/2 (1/10-3 d),1/2 (1/10-3 d),d,d,d,0,0,9/10}} has maximum value equal to {3.89943*10⁻⁶,{a->0.0198292,d->0.0198292}}

Point number 7 of type {{1,0,0,0,0,0,0,0,0},
{0,1/70,1/70,1/70,1/70,1/70,1/70,1/70,9/10}} has maximum value equal to $8.13946569881597 \times 10^{-6}$

Point number 8 of type {{1,0,0,0,0,0,0,0,0},
{0,0,1/60,1/60,1/60,1/60,1/60,1/60,9/10}} has maximum value equal to $5.015432098765432 \times 10^{-6}$

Point number 9 of type {{1,0,0,0,0,0,0,0,0},
{0,0,0,1/50,1/50,1/50,1/50,1/50,9/10}} has maximum value equal to 2.0736×10^{-6}

Max attained in point 1 with value 0.000015114926603711038`.

UPPER BOUND ON THE RATE

The bound on the rate for $b=9$, $k=8$ and $\epsilon=1/10$ is 0.056153665045097105
and $\eta_0=0.986716849491376$