Additional material for the paper: "Improved Bounds for (b, k)-hashing"

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Extended list of Maxima candidates

In the following two sections we report an extended list of all the maxima candidates for the partition based on maximum value and for the partition based on minimum value.

Partition $\{\breve{P}_b^i\}_{i=0,\dots,b}$

Maxima candidates for $\breve{M}_1$

1) $(\frac{1}{b}, \dots, \frac{1}{b}; \frac{1}{b}, \dots, \frac{1}{b})$;

2) $(0, \frac{1}{b-1}, \dots, \frac{1}{b-1}; \gamma, \delta, \dots, \delta)$
where $\gamma + (b-1)\delta = 1$;

3) $(0, \alpha, \dots, \alpha, \beta; \gamma, \delta, \dots, \delta, 0)$
where $(b-2)\alpha + \beta = 1$ and $\gamma + (b-2)\delta = 1$;

4) $(0, 0, \frac{1}{b-2}, \dots, \frac{1}{b-2}; \gamma, \gamma, \delta, \dots, \delta)$
where $2\gamma + (b-2)\delta = 1$;

5) $(0, 0, \alpha, \dots, \alpha, \beta; \gamma, \gamma, \delta, \dots, \delta, 0)$
where $(b-3)\alpha + 2\beta = 1$ and $2\gamma + (b-3)\delta = 1$;

6) $(0, 0, \alpha, \dots, \alpha, \beta, \beta; \gamma, \gamma, \delta, \dots, \delta, 0, 0)$
where $(b-4)\alpha + 2\beta = 1$ and $2\gamma + (b-4)\delta = 1$;

$\dots$) $\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$

$\frac{(b-j+1)(b-j+2)}{2}$) $(0, \dots, 0, \alpha, \dots, \alpha, \beta, \dots, \beta; \gamma, \dots, \gamma, \delta, \dots, \delta, 0, \dots, 0)$
where $(2j-b)\alpha + 2(b-j)\beta = 1$ and $2(b-j)\gamma + (2j-b)\delta = 1$;

Exactly one coordinate of p equals to $1-\epsilon$

1) $(1-\epsilon, \frac{\epsilon}{b-1}, \dots, \frac{\epsilon}{b-1}; 0, \frac{1}{b-1}, \dots, \frac{1}{b-1})$;

2) $(1-\epsilon, \frac{\epsilon}{b-2}, \dots, \frac{\epsilon}{b-2}, 0; 0, \gamma, \dots, \gamma, \delta)$
where $(b-2)\gamma + \delta = 1$;

3) $(1-\epsilon, \alpha, \beta, \dots, \beta; 0, 0, \frac{1}{b-2}, \dots, \frac{1}{b-2})$
where $\alpha + (b-2)\beta = \epsilon$;

4) $(1-\epsilon, \alpha, \beta, \dots, \beta, 0; 0, 0, \gamma, \dots, \gamma, \delta)$
where $\alpha + (b-3)\beta = \epsilon$ and $(b-3)\gamma + \delta = 1$;

5) $(1-\epsilon, \frac{\epsilon}{b-3}, \dots, \frac{\epsilon}{b-3}, 0, 0; 0, \gamma, \dots, \gamma, \delta, \delta)$
where $(b-3)\gamma + 2\delta = 1$;

6) $(1-\epsilon,\alpha,\beta,\dots,\beta,0,0;0,0,\gamma,\dots,\gamma,\delta,\delta)$
where $\alpha+(b-4)\beta=\epsilon$ and $(b-4)\gamma+2\delta=1$;

$\dots$) $\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$

$(b-j)(b-j+1)$) $(1-\epsilon,\alpha,\dots,\alpha,\beta,\dots,\beta,0,\dots,0;0,\dots,0,\gamma,\dots,\gamma,\delta,\dots,\delta)$
where $(b-j-1)\alpha+(2j-b)\beta=\epsilon$ and $(2j-b)\gamma+(b-j)\delta=1$;

$(b-j)(b-j+1)+1$) $(1-\epsilon,\epsilon,0,\dots,0,0;0,\gamma,\delta,\dots,\delta)$
where $\gamma+(b-2)\delta=1$;

$(b-j)(b-j+1)+2$) $(1-\epsilon,\epsilon,0,\dots,0;0,0,\frac{1}{b-2},\dots,\frac{1}{b-2})$;

$\dots$) $\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$

$(b-j)(b-j+1)+b-j$) $(1-\epsilon,\epsilon,0,\dots,0;0,\dots,0,\frac{1}{j},\dots,\frac{1}{j})$;

Exactly one coordinate of p and one coordinate of q equal to $1-\epsilon$

1) $(1-\epsilon,\frac{\epsilon}{b-2},\dots,\frac{\epsilon}{b-2},0;0,\frac{\epsilon}{b-2},\dots,\frac{\epsilon}{b-2},1-\epsilon)$;

2) $(1-\epsilon,\frac{\epsilon}{b-3},\dots,\frac{\epsilon}{b-3},0,0;0,\gamma,\dots,\gamma,\delta,1-\epsilon)$
where $(b-3)\gamma+\delta=\epsilon$;

3) $(1-\epsilon,\alpha,\beta,\dots,\beta,0,0;0,0,\gamma,\dots,\gamma,\delta,1-\epsilon)$
where $\alpha+(b-4)\beta=\epsilon$ and $(b-4)\gamma+\delta=\epsilon$;

$\dots$) $\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$

$\frac{(b-j)(b-j+1)}{2}$) $(1-\epsilon,\alpha\dots,\alpha,\beta,\dots,\beta,0,\dots,0;0,\dots,0,\gamma,\dots,\gamma,\delta\dots,\delta,1-\epsilon)$
where $(b-j-1)\alpha+(2j-b)\beta=\epsilon$ and $(2j-b)\gamma+(b-j-1)\delta=\epsilon$;

$\frac{(b-j)(b-j+1)}{2}+1$) $(1-\epsilon,\epsilon,0,\dots,0,0;0,\gamma,\delta,\dots,\delta,1-\epsilon)$
where $\gamma+(b-3)\delta=\epsilon$;

$\frac{(b-j)(b-j+1)}{2}+2$) $(1-\epsilon,\epsilon,0,\dots,0,0;0,0,\frac{\epsilon}{b-3},\dots,\frac{\epsilon}{b-3},1-\epsilon)$.

$\dots$) $\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$

$\frac{(b-j)(b-j+1)}{2}+b-j$) $(1-\epsilon,\epsilon,0,\dots,0,0;0,\dots,0,\frac{\epsilon}{j-1},\dots,\frac{\epsilon}{j-1},1-\epsilon)$.

Maxima candidates for $\breve{M}_2$

1) $(\frac{1}{b},\dots,\frac{1}{b};\frac{1}{b},\dots,\frac{1}{b})$;

2) $(0,\frac{1}{b-1},\dots,\frac{1}{b-1};\gamma,\delta,\dots,\delta)$
where $\gamma+(b-1)\delta=1$;

3) $(0,\alpha,\dots,\alpha,\beta;\gamma,\delta,\dots,\delta,0)$
where $(b-2)\alpha+\beta=1$ and $\gamma+(b-2)\delta=1$;

4) $(0,0,\frac{1}{b-2},\dots,\frac{1}{b-2};\gamma,\gamma,\delta,\dots,\delta)$
where $2\gamma+(b-2)\delta=1$;

5) $(0,0,\alpha,\dots,\alpha,\beta;\gamma,\gamma,\delta,\dots,\delta,0)$
where $(b-3)\alpha+2\beta=1$ and $2\gamma+(b-3)\delta=1$;

6) $(0,0,\alpha,\dots,\alpha,\beta,\beta;\gamma,\gamma,\delta,\dots,\delta,0,0)$
where $(b-4)\alpha+2\beta=1$ and $2\gamma+(b-4)\delta=1$;

$\dots$) $\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$

$\frac{(b-j+1)(b-j+2)}{2}$) $(0,\dots,0,\alpha,\dots,\alpha,\beta,\dots,\beta;\gamma,\dots,\gamma,\delta,\dots,\delta,0,\dots,0)$
where $(2j-b)\alpha+(b-j)\beta=1$ and $(b-j)\gamma+(2j-b)\delta=1$.

Maxima candidates for $\breve{M}_3$

1) $(1-\epsilon, \frac{\epsilon}{b-1}, \dots, \frac{\epsilon}{b-1}; 1-\epsilon, \frac{\epsilon}{b-1}, \dots, \frac{\epsilon}{b-1});$

2) $(1-\epsilon, \frac{\epsilon}{b-2}, \dots, \frac{\epsilon}{b-2}, 0; 1-\epsilon, \gamma, \dots, \gamma, \delta)$
where $(b-2)\gamma + \delta = \epsilon$;

3) $(1-\epsilon, \alpha, \dots, \alpha, \beta, 0; 1-\epsilon, \gamma, \dots, \gamma, 0, \delta)$
where $(b-3)\alpha + \beta = \epsilon$ and $(b-3)\gamma + \delta = \epsilon$;

4) $(1-\epsilon, \frac{\epsilon}{b-3}, \dots, \frac{\epsilon}{b-3}, 0, 0; 1-\epsilon, \gamma, \dots, \gamma, \delta, \delta)$
where $(b-3)\gamma + 2\delta = \epsilon$;

5) $(1-\epsilon, \alpha, \dots, \alpha, \beta, 0, 0; 1-\epsilon, \gamma, \dots, \gamma, 0, \delta, \delta)$
where $(b-4)\alpha + \beta = \epsilon$ and $(b-4)\gamma + 2\delta = \epsilon$;

6) $(1-\epsilon, \alpha, \dots, \alpha, \beta, \beta, 0, 0; 1-\epsilon, \gamma, \dots, \gamma, 0, 0, \delta, \delta)$
where $(b-5)\alpha + 2\beta = \epsilon$ and $(b-5)\gamma + 2\delta = \epsilon$;

$\dots)$ $\dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots$

$\frac{(b-j+1)(b-j+2)}{2})$ $(1-\epsilon, \alpha, \dots, \alpha, \beta, \dots, \beta, 0, \dots, 0; 1-\epsilon, \gamma, \dots, \gamma, 0, \dots, 0, \delta, \dots, \delta)$
where $(2j-b-1)\alpha + (b-j)\beta = \epsilon$ and $(2j-b-1)\gamma + (b-j)\delta = \epsilon$;

$\frac{(b-j+1)(b-j+2)}{2} + 1)$ $(1-\epsilon, \epsilon, 0, \dots, 0; 1-\epsilon, \gamma, \delta, \dots, \delta)$
where $\gamma + (b-2)\delta = \epsilon$;

$\frac{(b-j+1)(b-j+2)}{2} + 2)$ $(1-\epsilon, \epsilon, 0, \dots, 0; 1-\epsilon, 0, \frac{\epsilon}{b-2}, \dots, \frac{\epsilon}{b-2});$

$\frac{(b-j+1)(b-j+2)}{2} + 3)$ $(1-\epsilon, \epsilon, 0, 0, \dots, 0; 1-\epsilon, 0, 0, \frac{\epsilon}{b-3}, \dots, \frac{\epsilon}{b-3}).$

$\dots)$ $\dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots$

$\frac{(b-j+1)(b-j+2)}{2} + b - j + 1)$ $(1-\epsilon, \epsilon, 0, 0, \dots, 0; 1-\epsilon, 0, \dots, 0, \frac{\epsilon}{j-1}, \dots, \frac{\epsilon}{j-1}).$

Maxima candidates for $\breve{M}_4$

1) $(1-\epsilon+\sigma, \frac{\epsilon-\sigma}{b-2}, \dots, \frac{\epsilon-\sigma}{b-2}, 0; 0, \frac{\epsilon-\phi}{b-2}, \dots, \frac{\epsilon-\phi}{b-2}, 1-\epsilon+\phi)$
where $0 \le \sigma, \phi \le \epsilon$;

2) $(1-\epsilon+\sigma, \frac{\epsilon-\sigma}{b-3}, \dots, \frac{\epsilon-\sigma}{b-3}, 0, 0; 0, \gamma, \dots, \gamma, \delta, 1-\epsilon+\phi)$
where $0 \le \sigma, \phi \le \epsilon$ and $(b-3)\gamma + \delta = \epsilon - \phi$;

3) $(1-\epsilon+\sigma, \alpha, \beta, \dots, \beta, 0, 0; 0, 0, \gamma, \dots, \gamma, \delta, 1-\epsilon+\phi)$
where $0 \le \sigma, \phi \le \epsilon$, $\alpha + (b-4)\beta = \epsilon - \sigma$ and $(b-4)\gamma + \delta = \epsilon - \phi$;

$\dots)$ $\dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots$

$\frac{(b-j)(b-j+1)}{2})$ $(1-\epsilon+\sigma, \alpha, \dots, \alpha, \beta, \dots, \beta, 0, \dots, 0; 0, \dots, 0, \gamma, \dots, \gamma, \delta, \dots, \delta, 1-\epsilon+\phi)$
where $0 \le \sigma, \phi \le \epsilon$, $(b-j-1)\alpha + (2j-b)\beta = \epsilon - \sigma$ and $(2j-b)\gamma + (b-j-1)\delta = \epsilon - \phi$;

$\frac{(b-j)(b-j+1)}{2} + 1)$ $(1, 0, \dots, 0, 0; 0, \frac{\epsilon-\phi}{b-2}, \dots, \frac{\epsilon-\phi}{b-2}, 1-\epsilon+\phi)$
where $0 \le \phi \le \epsilon$;

$\frac{(b-j)(b-j+1)}{2} + 2)$ $(1, 0, 0, \dots, 0, 0; 0, 0, \frac{\epsilon-\phi}{b-3}, \dots, \frac{\epsilon-\phi}{b-3}, 1-\epsilon+\phi)$
where $0 \le \phi \le \epsilon$.

$\dots)$ $\dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots$

$\frac{(b-j)(b-j+1)}{2} + b - j)$ $(1, 0, 0, \dots, 0, 0; 0, \dots, 0, \frac{\epsilon-\phi}{j-1}, \dots, \frac{\epsilon-\phi}{j-1}, 1-\epsilon+\phi)$
where $0 \le \phi \le \epsilon$.

Partition $\{\widehat{P}_k^i\}_{i=0,\dots,k}$

Maxima candidates for $\widehat{M}_1$

1) $(\frac{1}{b}, \dots, \frac{1}{b}; \frac{1}{b}, \dots, \frac{1}{b})$;

2) $(\epsilon, \frac{1-\epsilon}{b-1}, \dots, \frac{1-\epsilon}{b-1}; \gamma, \delta, \dots, \delta)$
where $\gamma, \delta \geq \epsilon$ and $\gamma + (b-1)\delta = 1$;

3) $(\epsilon, \alpha, \dots, \alpha, \beta; \gamma, \delta, \dots, \delta, \epsilon)$
where $\alpha, \beta, \gamma, \delta \geq \epsilon$, $(b-2)\alpha + \beta + \epsilon = 1$ and $\gamma + (b-2)\delta + \epsilon = 1$;

4) $(\epsilon, \epsilon, \frac{1-\epsilon}{b-2}, \dots, \frac{1-\epsilon}{b-2}; \gamma, \gamma, \delta, \dots, \delta)$
where $\gamma, \delta \geq \epsilon$ and $2\gamma + (b-2)\delta = 1$;

5) $(\epsilon, \epsilon, \alpha, \dots, \alpha, \beta; \gamma, \gamma, \delta, \dots, \delta, \epsilon)$
where $\alpha, \beta, \gamma, \delta \geq \epsilon$, $(b-3)\alpha + \beta + 2\epsilon = 1$ and $2\gamma + (b-3)\delta + \epsilon = 1$;

6) $(\epsilon, \epsilon, \alpha, \dots, \alpha, \beta, \beta; \gamma, \gamma, \delta, \dots, \delta, \epsilon, \epsilon)$
where $\alpha, \beta, \gamma, \delta \geq \epsilon$, $(b-4)\alpha + 2\beta + 2\epsilon = 1$ and $2\gamma + (b-4)\delta + 2\epsilon = 1$;

$\dots$) $\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$

$\frac{b(b+1)}{2}$) $(\epsilon, \epsilon, \dots, \epsilon, 1-(b-1)\epsilon; 1-(b-1)\epsilon, \epsilon, \dots, \epsilon)$.

Maxima candidates for $\widehat{M}_2$

1) $(\alpha, \dots, \alpha, \beta; \frac{1-\epsilon}{b-1}, \dots, \frac{1-\epsilon}{b-1}, \epsilon)$
where $\alpha \leq \epsilon$ and $\beta + (b-1)\alpha = 1$;

2) $(\alpha, \dots, \alpha, \beta, \beta; \frac{1-\epsilon}{b-2}, \dots, \frac{1-\epsilon}{b-2}, \epsilon, \epsilon)$
where $\alpha \leq \epsilon$ and $2\beta + (b-2)\alpha = 1$;

$\dots$) $\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$

$b-1$) $(\alpha, \beta, \dots, \beta; 1-(b-1)\epsilon, \epsilon, \dots, \epsilon)$
where $\alpha \leq \epsilon$ and $\alpha + (b-1)\beta = 1$;

b) $(\epsilon, \frac{1-\epsilon}{b-1}, \dots, \frac{1-\epsilon}{b-1}; \gamma, \delta, \dots, \delta)$
where $\gamma, \delta \geq \epsilon$ and $\gamma + (b-1)\delta = 1$;

$b+1$) $(\epsilon, \alpha, \dots, \alpha, \beta; \gamma, \delta, \dots, \delta, \epsilon)$
where $\gamma, \delta \geq \epsilon$, $\beta + (b-2)\alpha = 1-\epsilon$ and $\gamma + (b-2)\delta = 1-\epsilon$;

$b+2$) $(\epsilon, \alpha, \dots, \alpha, \beta, \beta; \gamma, \delta, \dots, \delta, \epsilon, \epsilon)$
where $\gamma, \delta \geq \epsilon$, $2\beta + (b-3)\alpha = 1-\epsilon$ and $\gamma + (b-3)\delta = 1-2\epsilon$;

$\dots$) $\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$

$2b-1$) $(\epsilon, \frac{1-\epsilon}{b-1}, \dots, \frac{1-\epsilon}{b-1}; 1-(b-1)\epsilon, \epsilon, \dots, \epsilon)$;

$2b$) $(0, \frac{1}{b-1}, \dots, \frac{1}{b-1}; \gamma, \delta, \dots, \delta)$
where $\gamma, \delta \geq \epsilon$ and $\gamma + (k-1)\delta = 1$;

$2b+1$) $(0, \alpha, \dots, \alpha, \beta; \gamma, \delta, \dots, \delta, \epsilon)$
where $\gamma, \delta \geq \epsilon$, $\beta + (b-2)\alpha = 1$ and $\gamma + (b-2)\delta = 1-\epsilon$;

$2b+2$) $(0, \alpha, \dots, \alpha, \beta, \beta; \gamma, \delta, \dots, \delta, \epsilon, \epsilon)$
where $\gamma, \delta \geq \epsilon$, $2\beta + (b-3)\alpha = 1$ and $\gamma + (b-3)\delta = 1-2\epsilon$;

$\dots$) $\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$

$3b-1)$ $(0, \frac{1}{b-1}, \dots, \frac{1}{b-1}; 1-(b-1)\epsilon, \epsilon, \dots, \epsilon)$;

$3b)$ $(0, 0, \frac{1}{b-2}, \dots, \frac{1}{b-2}; \gamma, \gamma, \delta, \dots, \delta)$
where $\gamma, \delta \geq \epsilon$ and $2\gamma + (b-3)\delta = 1$;

$3b+1)$ $(0, 0, \alpha, \dots, \alpha, \beta; \gamma, \gamma, \delta, \dots, \delta, \epsilon)$
where $\gamma, \delta \geq \epsilon$, $\beta + (b-3)\alpha = 1$ and $2\gamma + (b-3)\delta = 1-\epsilon$;

$3b+2)$ $(0, 0, \alpha, \dots, \alpha, \beta, \beta; \gamma, \gamma, \delta, \dots, \delta, \epsilon, \epsilon)$
where $\gamma, \delta \geq \epsilon$, $2\beta + (b-4)\alpha = 1$ and $2\gamma + (b-4)\delta = 1-2\epsilon$;

$\dots)$ $\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$

$4b-1)$ $(0, 0, \frac{1}{b-2}, \dots, \frac{1}{b-2}; 1-(b-1)\epsilon, \epsilon, \dots, \epsilon)$;

$\dots)$ $\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$

$b^2-b(j-2)-1)$ $(0, \dots, 0, \frac{1}{j}, \dots, \frac{1}{j}; 1-(b-1)\epsilon, \epsilon, \dots, \epsilon)$;

$b^2-b(j-2))$ $(0, \dots, 0, 1; \gamma, \dots, \gamma, \delta)$
where $\gamma, \delta \geq \epsilon$ and $\delta + (b-1)\gamma = 1$;

$b^2-b(j-2)+1)$ $(0, \dots, 0, 1; \frac{1-\epsilon}{b-1}, \dots, \frac{1-\epsilon}{b-1}, \epsilon)$;

$b^2-b(j-2)+2)$ $(0, \dots, 0, 1; \frac{1-2\epsilon}{b-2}, \dots, \frac{1-2\epsilon}{b-2}, \epsilon, \epsilon)$;

$\dots)$ $\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$

$b^2-b(j-3)-1)$ $(0, \dots, 0, 1; 1-(b-1)\epsilon, \epsilon, \dots, \epsilon, \epsilon)$.

Maxima candidates for an upper bound on $\widehat{M}_3$

1) $(\sigma, \frac{1-\sigma}{b-1}, \dots, \frac{1-\sigma}{b-1}; \phi, \frac{1-\phi}{b-1}, \dots, \frac{1-\phi}{b-1})$
where $0 \leq \sigma, \phi \leq \epsilon$;

2) $(\sigma, \alpha, \dots, \alpha, \beta; \phi, \frac{1-\phi}{b-2}, \dots, \frac{1-\phi}{b-2}, 0)$
where $0 \leq \sigma, \phi \leq \epsilon$ and $\beta + (b-2)\alpha = 1-\sigma$;

3) $(\sigma, \alpha, \dots, \alpha, \beta, \beta; \phi, \frac{1-\phi}{b-3}, \dots, \frac{1-\phi}{b-3}, 0, 0)$
where $0 \leq \sigma, \phi \leq \epsilon$ and $2\beta + (b-3)\alpha = 1-\sigma$;

4) $(\sigma, 0, \alpha, \dots, \alpha, \beta; \phi, \gamma, \delta, \dots, \delta, 0)$
where $\sigma, \phi \leq \epsilon$, $\sigma + \beta + (b-3)\alpha = 1$ and $\phi + \gamma + (b-3)\delta = 1$;

5) $(\sigma, 0, \alpha, \dots, \alpha, \beta, \beta; \phi, \gamma, \delta, \dots, \delta, 0, 0)$
where $0 \leq \sigma, \phi \leq \epsilon$, $\sigma + 2\beta + (b-4)\alpha = 1$ and $\phi + \gamma + (b-4)\delta = 1$;

6) $(\sigma, 0, 0, \alpha, \dots, \alpha, \beta, \beta; \phi, \gamma, \gamma, \delta, \dots, \delta, 0, 0)$
where $0 \leq \sigma, \phi \leq \epsilon$, $\sigma + 2\beta + (b-5)\alpha = 1$ and $\phi + 2\gamma + (b-5)\delta = 1$;

$\dots)$ $\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$

$\frac{(b-j+1)(b-j+2)}{2})$ $(\sigma, 0, \dots, 0, \alpha, \dots, \alpha, \beta, \dots, \beta; \phi, \gamma, \dots, \gamma, \delta, \dots, \delta, 0, \dots, 0)$
where $0 \leq \sigma, \phi \leq \epsilon$, $\sigma + (b-j)\beta + (2j-b-1)\alpha = 1$ and $\phi + (b-j)\gamma + (2j-b-1)\delta = 1$;

$\frac{(b-j+1)(b-j+2)}{2}+1)$ $(\sigma, 0, \frac{1-\sigma}{b-2}, \dots, \frac{1-\sigma}{b-2}; 0, 1, 0, \dots, 0)$
where $0 \leq \sigma \leq \epsilon$;

$\frac{(b-j+1)(b-j+2)}{2}+2)$ $(\sigma, 0, 0, \frac{1-\sigma}{b-3}, \dots, \frac{1-\sigma}{b-3}; 0, 1, 0, 0, \dots, 0)$
where $0 \leq \sigma \leq \epsilon$;

$\dots)$ $\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$

$\frac{(b-j+1)(b-j+2)}{2} + b - j)$ $(\sigma, 0, \dots, 0, \frac{1-\sigma}{j-1}, \dots, \frac{1-\sigma}{j-1}; 0, 1, 0, \dots, 0)$
where $0 \le \sigma \le \epsilon$.

Maxima candidates for $\widehat{M}_4$

1) $(\frac{1}{b}, \dots, \frac{1}{b}; \frac{1}{b}, \dots, \frac{1}{b})$;

2) $(0, \frac{1}{b-1}, \dots, \frac{1}{b-1}; \gamma, \delta, \dots, \delta)$
where $\gamma + (b-1)\delta = 1$;

3) $(0, \alpha, \dots, \alpha, \beta; \gamma, \delta, \dots, \delta, 0)$
where $(b-2)\alpha + \beta = 1$ and $\gamma + (b-2)\delta = 1$;

4) $(0, 0, \frac{1}{b-2}, \dots, \frac{1}{b-2}; \gamma, \gamma, \delta, \dots, \delta)$
where $2\gamma + (b-2)\delta = 1$;

5) $(0, 0, \alpha, \dots, \alpha, \beta; \gamma, \gamma, \delta, \dots, \delta, 0)$
where $(b-3)\alpha + 2\beta = 1$ and $2\gamma + (b-3)\delta = 1$;

6) $(0, 0, \alpha, \dots, \alpha, \beta, \beta; \gamma, \gamma, \delta, \dots, \delta, 0, 0)$
where $(b-4)\alpha + 2\beta = 1$ and $2\gamma + (b-4)\delta = 1$;

$\dots)$ $\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$

$\frac{(b-j+1)(b-j+2)}{2})$ $(0, \dots, 0, \alpha, \dots, \alpha, \beta, \dots, \beta; \gamma, \dots, \gamma, \delta, \dots, \delta, 0, \dots, 0)$
where $(2j-b)\alpha + (b-j)\beta = 1$ and $(b-j)\gamma + (2j-b)\delta = 1$.