The Probabilistic Method applied to Graphs

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Introduction

It is a powerful tool to attack many problems in discrete mathematics.

The main idea is trying to prove that a structure with certain properties exists there are two main steps:

1. define an appropriate probability space of structures;
2. show that the desired properties hold in this space with positive probability.

Introduction

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The main idea is trying to prove that a structure with certain properties exists there are two main steps:

1. define an appropriate probability space of structures;
2. show that the desired properties hold in this space with positive probability.

The method is best illustrated by examples

Ramsey-number

The Ramsey-number $R(k, l)$ is the smallest integer n such that in any 2-coloring of the edges of a complete graph on n vertices K_n by red and blue, either there is a red K_k or a blue K_l. Ramsey in 1929 showed that $R(k, l)$ is finite for any two integers k and l.

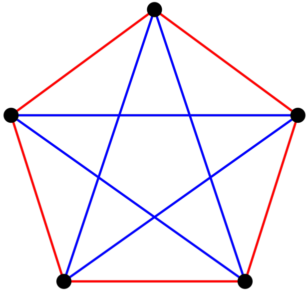

Figure: *Example of a 2-coloring in which there is no K_3 monochromatic, i.e.,* $R(3, 3) > 5$

Ramsey-number 1

Theorem

If $\binom{n}{k}2^{1-\binom{k}{2}} < 1$ *then* $R(k,k) > n$*. Thus* $R(k,k) > \lfloor 2^{\frac{k}{2}} \rfloor$ *for all* $k \geq 3$*.*

Proof.

Consider a random 2-coloring of the edges of K_n. For any fix set R of k vertices, let A_R be the event that the induced subgraph of K_n on R is monochromatic. Clearly $P(A_R) = 2^{1-\binom{k}{2}}$ and $P(\cup_R A_R) \leq \binom{n}{k}2^{1-\binom{k}{2}}$. If $\binom{n}{k}2^{1-\binom{k}{2}} < 1$ then with positive probability no event A_R occurs, i.e., $R(k,k) > n$. Take $n = \lfloor 2^{\frac{k}{2}} \rfloor$. □

Best $n \sim \frac{1}{e\sqrt{2}}(1 + o(1))k2^{k/2}$.

Ramsey-number 2

Theorem

For any integer n

$$R(k,k) > n - \binom{n}{k} 2^{1-\binom{k}{2}}$$

Proof.

Consider a random 2-coloring of the edges of K_n. For any set R of size k let X_R be the indicator random variable for the event that the induced subgraph of K_n on R is monochromatic. Let $X = \sum_R X_R$. $E[X] = \binom{n}{k} 2^{1-\binom{k}{2}}$. There exists a 2-coloring for which $X \leq E[X]$, fix such a coloring. Remove from each monochromatic k-set one vertex. □

Best $n \sim \frac{1}{e}(1 - o(1))k2^{k/2}$ that gives $R(k,k) > \frac{1}{e}(1 + o(1))k2^{k/2}$.

Lovász Local Lemma

Lemma (Lovász 1975)

Let $A_1, A_2, \ldots, A_n$ be events in arbitrary probability space. A directed graph $D = (V, E)$ is called dependency graph for the events $A_1, A_2, \ldots, A_n$ if for each i, the event A_i is mutually independent of all events $\{A_j : (i,j) \notin E\}$.

Suppose that there exists real numbers $x_1, x_2, \ldots, x_n$ such that $0 \le x_i < 1$ and $P(A_i) \le x_i \prod_{(i,j)\in E}(1 - x_j)$ for all $1 \le i \le n$. Then

$$P(\cap_{i=1}^{n} \overline{A}_i) \ge \prod_{i=1}^{n}(1 - x_i).$$

Lovász Local Lemma - Proof

Induction on $S \subset \{1, 2, \dots, n\}$, $|S| = s < n$ and any $i \notin S$,

$$P(A_i | \cap_{j \in S} \overline{A}_j) \leq x_i.$$

Clearly true for $s = 0$. Assuming it holds for $s' < s$.
Put $S_1 = \{j \in S : (i, j) \in E\}$ and $S_2 = S \setminus S_1$. Then

$$P(A_i | \cap_{j \in S} \overline{A}_j) = \frac{P(A_i \cap (\cap_{j \in S_1} \overline{A}_j) \mid \cap_{k \in S_2} \overline{A}_k)}{P(\cap_{j \in S_1} \overline{A}_j | \cap_{k \in S_2} \overline{A}_k)}$$

the numerator can be easily upper bound
$P(A_i \cap (\cap_{j \in S_1} \overline{A}_j) \mid \cap_{k \in S_2} \overline{A}_k) \leq P(A_i | \cap_{k \in S_2} \overline{A}_k) = P(A_i) \leq x_i \prod_{(i,j) \in E} (1 - x_j)$.
The denominator can be lower bounded thanks to the induction hypothesis, suppose $S_1 = (j_1, j_2, \dots, j_r)$

$$P(\cap_{h=1}^{r} \overline{A}_{j_h} | \cap_{k \in S_2} \overline{A}_k) = \prod_{h=1}^{r} (1 - P(A_{j_h} | (\cap_{l=1}^{h-1} \overline{A}_{j_l}) \cap (\cap_{k \in S_2} \overline{A}_k))) \geq \prod_{(i,j) \in E}^{n} (1 - x_j)$$

Lovász Local Lemma - Symmetric Case

Let $A_1, A_2, \ldots, A_n$ be events in arbitrary probability space. Suppose that each event A_i is dependent of a set of at most d events A_j, and that $P(A_i) \leq p$ for all $1 \leq i \leq n$. If

$$ep(d+1) \leq 1$$

then $P(\cap_{i=1}^{n} \overline{A}_i) > 0$.

Proof.

If $d = 0$ then is trivial. Otherwise, we can apply the Lovász Local Lemma taking $x_i = \frac{1}{d+1} < 1$ for $i = 1, 2, \ldots, n$ and using the fact that $\left(1 - \frac{1}{d+1}\right)^d > \frac{1}{e}$. □

Ramsey Numbers - Again

Consider the diagonal Ramsey number $R(k,k)$ and consider the random 2-coloring of the edges of K_n. For each set S of size k, $P(A_S) = 2^{1-\binom{k}{2}}$ is the probability that the subgraph is monochromatic. Each event A_S is not mutually independent of all events A_T for which $|S \cap T| \geq 2$.

Applying the symmetric Local Lemma with $p = 2^{1-\binom{k}{2}}$ and $d = \binom{k}{2}\binom{n}{k-2}$ we have that:

Theorem

If $e\left(\binom{k}{2}\binom{n}{k-2}+1\right)2^{1-\binom{k}{2}} < 1$ then $R(k,k) > n$. Some analysis shows that we should take n as follows

$$R(k,k) > \frac{\sqrt{2}}{e}(1+o(1))k2^{k/2}$$

Hypergraphs

Generalization of a graph in which an edge can join an arbitrary number of vertices while in an ordinary graph an edge connects two vertices.
$H = (V, E)$ where V is the set of vertices and E is the set of hyper-edges.

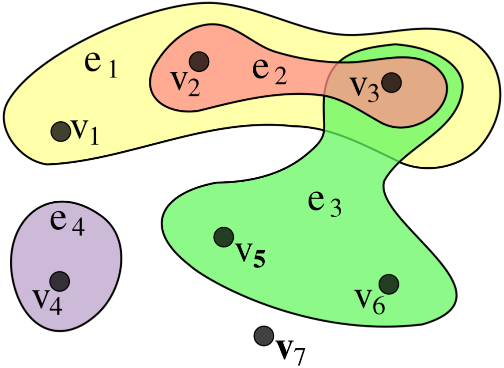

Figure: *Example of a hypergraph with* $|V| = 7$ *and* $|E| = 4$

n-uniform hypergraphs

Theorem (Erdős 1963)

Every n-uniform hypergraph with less than 2^{n-1} edges is 2-colorable, i.e, there exists a 2-coloring of V such that no edge is monochromatic.

Proof.

Let $H = (V, E)$ be a n-uniform hypergraph with $|E| < 2^{n-1}$ and consider a random 2-coloring of V. For each edge $e \in E$, let A_e be the event that e is monochromatic. Then

$$P(\cup_{e \in E} A_e) \leq \sum_{e \in E} P(A_e) = |E| 2^{1-n} < 1$$

so there exists a 2-coloring without monochromatic edges. □

Independent Sets

Theorem

Let $G = (V, E)$ have n vertices and $nd/2$ edges, $d \geq 1$. Then $\alpha(G) \geq \frac{n}{2d}$

Proof.

Let $S \subseteq V$ be a random subset defined by $Pr[v \in S] = p$. Let $X = |S|$ and Y be the number of edges in $G|_S$. For each $e = \{i, j\} \in E$ let Y_e be the indicator random variable for the event $i, j \in S$. So, $Y = \sum_e Y_e$. Then

$$E[Y_e] = Pr[i, j \in S] = p^2 \rightarrow E[Y] = nd/2p^2$$

Clearly $E[X] = np$, then

$$E[X - Y] = np - nd/2p^2$$

Set $p = 1/d$ in order to maximize $E[X - Y]$. □

High Girth and Chromatic Number - 1

$girth(G)$ = size of the shortest cycle

Theorem (Erdős 1959)

For all k, l there exists a graph G with $girth(G) > l$ and $\chi(G) > k$.

Fix $\theta < 1/l$ and let $G \sim G(n,p)$ with $p = n^{\theta-1}$. Let X be the number of cycles of size at most l. Then

$$E[X] = \sum_{i=3}^{l} \frac{(n)_i}{2i} p^i \leq \sum_{i=3}^{l} \frac{n^{\theta i}}{2i} = o(n)$$

In particular $Pr[X \geq n/2] = o(1)$ and setting $x = \lceil 3/p \log n \rceil$ we have

$$Pr[\alpha(G) \geq x] \leq \binom{n}{x} (1-p)^{\binom{x}{2}} < \left(ne^{-p(x-1)/2} \right)^x = o(1)$$

High Girth and Chromatic Number - 2

Let n be sufficiently large so that both events have probability less than $1/2$, there exists a graph G with less than $n/2$ cycles of length at most l and $\alpha(G) < 3n^{1-\theta} \log n$. Remove from G a vertex from each cycle of length at most l. This gives a graph G^* with $|G^*| \geq n/2$ and $girth(G^*) > l$. Clearly $\alpha(G^*) \leq \alpha(G)$ then

$$\chi(G^*) \geq \frac{|G^*|}{\alpha(G^*)} \geq \frac{n/2}{\alpha(G)} \geq \frac{n/2}{3n^{1-\theta} \log n} = \frac{n^{\theta}}{6 \log n}$$

Take a sufficiently large n so that the last term is greater than k.

Tournament

A tournament on a set V of n players is an orientation $T = (V, E)$ of edges of the complete graph.

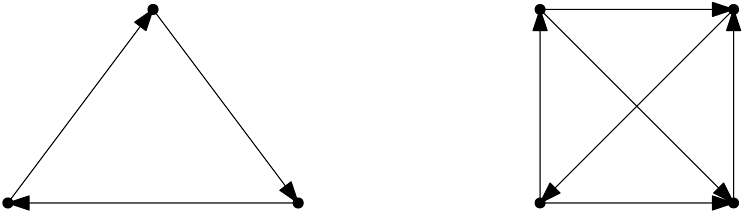

Figure: *Two tournaments when* $|V| = 3$ *and* $|V| = 4$

We say that a player v beats a player w if the edge (v, w) belongs to the edge-set.

Tournament - Theorem

A tournament has the property S_k if for every set of k players there is one who beats them all.

Theorem

If $\binom{n}{k}(1-2^{-k})^{n-k} < 1$ then there is a tournament on n vertices that has the property S_k.

Proof.

Consider a random tournament, for every fixed subset K of size k, let A_K be the event that there is no vertex which beats all the members of K. Clearly, $P(A_K) = (1-2^{-k})^{n-k}$ and $P(\cup_K A_K) \leq \binom{n}{k}(1-2^{-k})^{n-k}$. If $\binom{n}{k}(1-2^{-k})^{n-k} < 1$ then with positive probability no event A_K occurs. □

Tournament Szele - 1

Theorem (Szele (1943))

There is a tournament T with n players and at least $n!2^{-(n-1)}$ Hamiltonian Paths.

Proof.

In the random tournament, let X be the number of Hamiltonian paths. For each permutation σ, let X_σ be the indicator random variable for σ giving a Hamiltonian path, i.e., $(\sigma(i), \sigma(i+1)) \in T$ for $1 \leq i < n$. Then $X = \sum_\sigma X_\sigma$ and

$$E[X] = \sum_\sigma E[X_\sigma] = n!2^{-(n-1)}.$$

Hence some tournament has at least $E[X]$ Hamiltonian paths. □

Tournament Szele - 2

Problem

What is the maximum possible number of directed Hamiltonian paths in a tournament on n vertices?

Call this number $P(n)$. Szele shows that the following limit exists

$$\frac{1}{2} \leq \lim_{n\to\infty} \left(\frac{P(n)}{n!}\right)^{\frac{1}{n}} \leq \frac{1}{2^{3/4}}$$

and conjectures that the exact value is $1/2$.

Tournament Alon - 1

Theorem (Alon (1990))

There exists a positive constant c such that for every n

$$P(n) \leq cn^{\frac{3}{2}} \frac{n!}{2^{n-1}}$$

For a tournament T, denote by $P(T)$ the number of directed Hamiltonian paths in T, $C(T)$ the number of Hamiltonian cycles in T while $F(T)$ the number of spanning subgraphs of T in which each vertex has indegree and outdegree equal to 1.

Clearly, $C(T) \leq F(T)$.

Tournament Alon - 2

If $T = (V, E)$ is a tournament on a set of n vertices, let A_T be the adjacency matrix of T ($a_{ij} = 1$ if $(i, j) \in E$, $a_{ij} = 0$ otherwise) and let r_i denote the number of ones in row i of A_T. Clearly,

$$\sum_{i=1}^{n} r_i = \binom{n}{2}$$

Interpreting combinatorially the terms in the expansion of the permanent of $A_T = (a_{ij})$ we have that

$$F(T) = per(A_T) = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i\sigma(i)}$$

Tournament Alon - Proof

The numbers r_i are the outdegrees of the vertices of T. So, if exists one $r_i = 0$ then $F(T) = C(T) = 0$. Otherwise we can apply Brégman's Theorem that states:

Theorem (Brégman)

Let $A = (a_{ij})$ be a $n \times n$ matrix with all $a_{ij} \in \{0, 1\}$. Let r_i be the number of ones in the i-th row. We have that

$$per(A) \leq \prod_{i=1}^{n} (r_i!)^{1/r_i}.$$

Can be proved that the maximum of the right hand side is achieved when all r_i are "equal", i.e., if n is odd $r_i = \binom{n}{2}/n = \frac{n-1}{2}$.

Tournament Alon - Proof 1

Asymptotically the Stirling's formula gives

Proposition

For every tournament T on n vertices

$$C(T) \leq F(T) \leq (1 + o(1)) \frac{\sqrt{\pi}}{\sqrt{2e}} n^{3/2} \frac{(n-1)!}{2^n}$$

To complete the proof, given a tournament S on n vertices and let T be a random tournament obtained from S by adding a new vertex x and picking each oriented edge connecting x to all the other vertices randomly and independently.

Tournament Alon - Proof 2

For every fixed Hamiltonian path in S, the probability that it can be extended to an Hamiltonian cycle in T is exactly $1/4$.

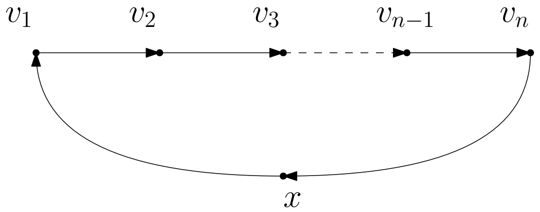

Figure: $v_1, v_2, \ldots, v_n$ *is an Hamiltonian path in* S *that is extended to a cycle in* T

Thus the expected cycles in T are $\frac{1}{4}P(S)$, so, there exists a T for which $C(T) \geq \frac{1}{4}P(S)$.

Tournament Alon - Proof 3

Thanks to the previous Proposition

$$C(T) \leq F(T) \leq (1 + o(1)) \frac{\sqrt{\pi}}{\sqrt{2}e}(n+1)^{3/2} \frac{n!}{2^{n+1}},$$

and we know that $P(S) \leq 4C(T)$, then

$$P(S) \leq O\left(n^{3/2} \frac{n!}{2^{n-1}}\right).$$

This completes the proof of the theorem.

References

N. Alon and J.H. Spencer, "The Probabilistic Method Second Edition", *2000*.