Graham's rearrangement for a class of semidirect products

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Graham's rearrangement conjecture

Let A be a finite subset of an abelian group.

An ordering $a_1, a_2, \ldots, a_{|A|}$ of A is *valid* if the partial sums

$$a_1, a_1 + a_2, \ldots, a_1 + a_2 + \ldots + a_{|A|}$$

are all distinct. Moreover, this ordering is called a *sequencing* if it is valid and $a_i + \ldots + a_j \neq 0$ for any $(i, j) \neq (1, |A|)$.

Conjecture (Graham (1971); Erdős and Graham (1980))

Let p be a prime. Then every subset $A \subseteq \mathbb{Z}_p \setminus \{0\}$ has a valid ordering.

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Example

Let $p = 5$ and $A = \{1, 2, 3, 4\} \subset \mathbb{Z}_5$. The ordering $1, 2, 3, 4$ is not valid since $1 = 1 + 2 + 3$ while the ordering $1, 3, 4, 2$ is valid since the partial sums are $1, 4, 3, 0$.

The only valid orderings of A beginning with 1 are $1, 3, 4, 2$ and $1, 2, 4, 3$ (with partial sums $1, 3, 2, 0$).

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Some motivation

This problem has connections to:

- Heffter arrays
- Graph decomposition
- Rainbow paths

In these contexts variants of this conjecture had been proposed by several authors. We recall here the following:

Conjecture (Alspach (∼2001))

Let G be an abelian group. Then every finite subset $A \subseteq G \setminus \{0\}$ has a valid ordering (a sequencing).

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Old and new results over $\mathbb{Z}_p$

Progress before 2024:

- Many authors (2005–2020):
  - Hicks, Ollis and Schmitt (2018): every $A \subseteq \mathbb{Z}_p \setminus \{0\}$ with $|A| \leq 10$ has a valid ordering.
  - Costa, D. F., Ollis and R–Frydman (2022): $|A| = 11, 12$ also works.
  - Both works use the **Combinatorial Nullstellensatz**.
- Bedert, Bucić, Kravitz, Montgomery, Müyesser (2025+): The conjecture holds for $|A| \geq p^{1-c}$. Previously it was proved for $|A| \geq p - 3$.

Theorem (N. Kravitz (July 2024) – W. Sawin (2015))

Let p be a prime. Then every subset $A \subseteq \mathbb{Z}_p \setminus \{0\}$ of size $|A| \leq \log p / 2 \log \log p$ has a valid ordering (a sequencing).

Theorem (B. Bedert and N. Kravitz (Sept. 2024))

Let p be a large prime. Then every subset $A \subseteq \mathbb{Z}_p \setminus \{0\}$ of size $|A| \leq e^{c(\log p)^{1/4}}$ has a valid ordering (a sequencing).

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Proof sketch.

Rectification step: The Pigeonhole Principle provides a $\lambda \in \mathbb{Z}_p \setminus \{0\}$ such that $\lambda \cdot A$ is contained in $(-p/4|A|, p/4|A|)$. Since the subset sums in $\lambda \cdot A$ have no "wrap-around" we can work in $\mathbb{Z}$.

By induction on $|A|$, one can prove that every finite subset $A \subseteq \mathbb{Z} \setminus \{0\}$ has a valid ordering.

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New Idea.

A set A to be not rectificable has to contain a large "dissociated set". A dissociated set D, $|D| = r$, is a set in which all the 2^r subset sums are distinct.

The dimension $dim(B)$ of a set B is the maximum size of a dissociated subset of B.

Theorem

Let $A \subseteq \mathbb{Z}_p$. Then $A = \cup_{j=1}^{s} D_j \cup E$ where each $|D_j|$ has size $\Theta(R)$ and E is rectifiable (i.e. $\dim(E) < R$).

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Finding the orderings - Probabilistic method

Goal: Find a valid ordering consisting of the positive elements of E, then the elements of the dissociated sets D_j's and then the negative elements of E.

- Randomly split the dissociated sets and then permute the newly splitted sets
- Inductively order the set E
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Our result

We denote by Dih_p the dihedral group

$$Dih_p = \mathbb{Z}_p \rtimes_\varphi \mathbb{Z}_2$$

with group operation

$$(x_1, a_1) \cdot (x_2, a_2) = (x_1 + \varphi_{a_1} x_2,\, a_1 + a_2),$$

where $\varphi_0 = 1$ and $\varphi_1 = -1$.

Inspired by the procedure of Bedert and Kravitz we get

Theorem (Costa, D. F., and Engel (2025+))

Let p be a large enough prime and $c > 0$. Then every subset $A \subseteq Dih_p \setminus \{id\}$ admits a valid ordering (a sequencing) provided that

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Proof sketch of our result

1. **Rectification:** If $\dim(B) < R$, there exists an automorphism $\phi \in \mathrm{Aut}(Dih_p)$ such that the projection $\pi_1(\phi(B))$ lies in a short interval of $\mathbb{Z}_p$ (i.e we can work in $Dih(\mathbb{Z})$).
2. **Structure Theorem:** Every subset A can be decomposed as
$$A = E \cup \bigcup_{j=1}^{s} D_j,$$
where E is **rectifiable** ($\pi_1(E)$ can be placed inside a short interval) and each D_j is a dissociated set of size $\Theta(R)$.
3. **Key property of D_j:** All elements of a given D_j share the same $\mathbb{Z}_2$-projection. The total product of D_j is invariant under reorderings with respect to the positions parity.
This allows us to define
$$\delta = \prod_{j=1}^{s} \prod_{d \in D_j} d,$$
independently of such reorderings.

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Idea for ordering $E \cup \{\delta\}$:

- By rectification, we may view $E \cup \{\delta\}$ as a subset of $Dih(\mathbb{Z})$.
- Decompose $E \cup \{\delta\}$ into three sets:

$$P = \{x \in E \cup \{\delta\} : \pi_2(x) = 0,\ \pi_1(x) > 0\},$$

$$N = \{x \in E \cup \{\delta\} : \pi_2(x) = 0,\ \pi_1(x) < 0\},$$

$$S = \{x \in E \cup \{\delta\} : \pi_2(x) = 1\}.$$

- Split S into S_e and S_o so that:

$$|S_e| = \lceil |S|/2 \rceil, \quad |S_o| = \lfloor |S|/2 \rfloor,$$

and the first components of S_e are all larger than those of S_o.

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Valid ordering for $E \cup \{\delta\}$:

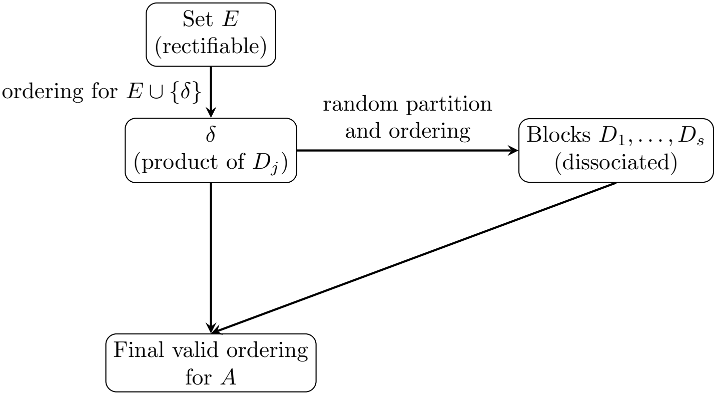
- Choose $s_0 \in S_e$.
- Any ordering of the form

$$\mathbf{p}\ s_0\ \mathbf{n}\ \mathbf{s}$$

  is a valid ordering for $E \cup \{\delta\}$, where:
  - $\mathbf{p}$ is an ordering of P,
  - $\mathbf{n}$ is an ordering of N,
  - $\mathbf{s}$ is an ordering of $(S_e \cup S_o) \setminus \{s_0\}$ alternating elements of S_e and S_o.

This ensures all partial products are distinct and yields a valid ordering for $E \cup \{\delta\}$.

Proof sketch: Flow Diagram



Semidirect products

Theorem (Costa, D. F., and Engel (2025+))

Let H be a finite abelian group and each subset of H can be ordered such that all of its partial sums are distinct. There exists a $c > 0$ such that every subset $A \subseteq \mathbb{Z}_p \rtimes_\varphi H \setminus \{id\}$, where $\varphi : H \to Aut(\mathbb{Z}_p)$ satisfies $\varphi(h) \in \{id, -id\}$ for each $h \in H$, of size

$$|A| \leq e^{c(\log p)^{1/4}},$$

has a sequencing.

Thank you.