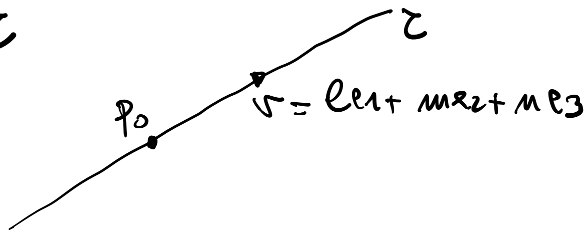
Rette in $A_3(\mathbb{R})$

Eq. parametrica

$$r: \begin{cases} x = x_0 + \ell t \\ y = y_0 + m t \\ z = z_0 + n t \end{cases} \quad t \in \mathbb{R} \qquad (\ell, m, n) \neq (0,0,0)$$

$$\text{pdr} = [(\ell, m, n)]$$

$$P_0 = (x_0, y_0, z_0) \in r$$



Eq. cartesiana

$$r: \begin{cases} ax + by + cz + d = 0 \\ a'x + b'y + c'z + d' = 0 \end{cases} \qquad \text{rk}\begin{pmatrix} a & b & c \\ a' & b' & c' \end{pmatrix} = 2$$

$$\infty^{3-2} = \infty^1 \text{ sol. (punti)}$$

Regola dei minori

$$\text{pdr} = [(\ell, m, n)]$$

$$\ell = \det\begin{pmatrix} b & c \\ b' & c' \end{pmatrix} \qquad m = -\det\begin{pmatrix} a & c \\ a' & c' \end{pmatrix} \qquad n = \det\begin{pmatrix} a & b \\ a' & b' \end{pmatrix}$$

Esempio

Determinare eq. cartesiane retta

1) per $P = (0,1,-1)$ con direzione $\overbrace{[(2,-2,6)]}^{\text{pdr}} = [(1,-1,3)]$

$$r: \begin{cases} x = 0 + t \\ y = 1 - t \\ z = -1 + 3t \end{cases} \quad t \in \mathbb{R} \qquad \begin{cases} t = x \\ y = 1 - x \\ z = -1 + 3x \end{cases} \Rightarrow r: \begin{cases} x + y - 1 = 0 \\ 3x - z - 1 = 0 \end{cases}$$

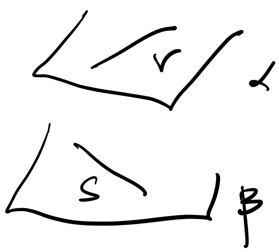
2) Per $H=(2,-2,1)$ e $K=(3,0,0)$

$\overline{HK} = K - H = (1,2,-1)$ $\quad$ pdr $= [(1,2,-1)]$

$$\begin{cases} x = 3+t \\ y = 2t \\ z = -t \end{cases} \quad t\in\mathbb{R} \Rightarrow \begin{cases} x = 3 - z \\ y = -2z \\ t = -z \end{cases} \Rightarrow r: \begin{cases} x+z-3=0 \\ y+2z=0 \end{cases}$$

Mutua posizione rette $A_3(\mathbb{R})$

r, s
- complanari
  - incidenti
  - parallele
    - coincidenti
    - distinte
- non complanari $\Rightarrow$ sghembe



$\exists\, \alpha, \beta$ piani paralleli tali che $r \subseteq \alpha$ e $s \subseteq \beta$.

ESEMPIO Determinare la posizione delle rette

$$r: \begin{cases} 2x+y-z=0 \\ 4x-z-19=0 \end{cases} \qquad s: \begin{cases} x-z-3=0 \\ y+4z-1=0 \end{cases}$$

$r \cap s \to$ studio SISTEMA 4 eq. in 3 incognite.

$$\begin{cases} 2x+y-z=0 \\ 4x-z-19=0 \\ x-z-3=0 \\ y+4z-1=0 \end{cases} \qquad A|B = \left(\begin{array}{ccc|c} 2 & 1 & -1 & 0 \\ 4 & 0 & -1 & 19 \\ 1 & 0 & -1 & 3 \\ 0 & 1 & 4 & 1 \end{array}\right)$$

$$\det\begin{pmatrix} 2 & 1 & -1 \\ 4 & 0 & -1 \\ 1 & 0 & -1 \end{pmatrix} = -\det\begin{pmatrix} 4 & -1 \\ 1 & -1 \end{pmatrix} \neq 0$$

- Se $\det(A|B) \neq 0 \Rightarrow$ SGHEMBE
- Se $\det(A|B) = 0$

incidenti

$\hookrightarrow \rho(A|B) = \rho(A) = 3 \Rightarrow \exists!$ sol. $\Rightarrow r \cap s = \{P\}$

$\hookrightarrow \rho(A) = 2, \rho(A|B) = 3 \Rightarrow \nexists$ sol. $r \parallel s$ e $r \neq s$

$\hookrightarrow \rho(A) = \rho(A|B) = 2 \Rightarrow \infty^1$ sol. $r \parallel s$ $r = s$

CALCOLO
$$\det(A|B) = \det\begin{pmatrix} 2 & 1 & -1 & 0 \\ 4 & 0 & -1 & 19 \\ 1 & 0 & -1 & 3 \\ -2 & 0 & 5 & 1 \end{pmatrix} \underset{R_4-R_1}{=} -\det\begin{pmatrix} 4 & -1 & 19 \\ 1 & -1 & 3 \\ -2 & 5 & 1 \end{pmatrix}$$

$$= -\det\begin{pmatrix} 0 & 3 & 7 \\ 1 & -1 & 3 \\ 0 & 3 & 7 \end{pmatrix} \begin{matrix} R_1-4R_2 \\ \\ R_3+2R_2 \end{matrix} = -\det\begin{pmatrix} 3 & 7 \\ 3 & 7 \end{pmatrix} = 0$$

$rk(A) = 3 = rk(A|B) \Rightarrow \infty^0$ sol. $\Rightarrow$ r e s INCIDENTI

Es. Al variare di $k \in \mathbb{R}$ si determini la posizione delle rette

$$r: \begin{cases} x + kz - 1 = 0 \\ y - 2 = 0 \end{cases} \qquad s: \begin{cases} z = 0 \\ 2x - 3y + \cancel{2z} - k = 0 \end{cases}$$

($2z = 0$)

$$A|B = \left(\begin{array}{ccc|c} 1 & 0 & k & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 2 & -3 & 0 & k \end{array}\right)$$

$$\det(A|B) = \det\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 2 & -3 & k \end{pmatrix} = k - 2 + 6 = k + 4$$

$h \neq -4 \Rightarrow r$ e s <u>SGHEMBE</u>

$h=-4$

$$A|B = \left(\begin{array}{ccc|c} \boxed{\begin{matrix}1 & 0 & -4\\ 0 & 1 & 0\\ 0 & 0 & 1\end{matrix}} & & & \begin{matrix}1\\2\\0\end{matrix} \\ 2 & -3 & 0 & -4 \end{array}\right) \qquad \det(\cdot) \neq 0 \qquad rk(A|B)=3$$

$\exists!$ unica soluzione, rette incidenti in 1 punto

$$\text{SPE}\ \begin{cases} x-4z=1 \\ y=2 \\ z=0 \end{cases} \Rightarrow \begin{cases} x=1 \\ y=2 \\ z=0 \end{cases} \Rightarrow P=(1,2,0)$$

PIANI IN $A_3(\mathbb{R})$

EQ. PARAMETRICA

$$\alpha: \begin{cases} x = x_0 + lt + l't' \\ y = y_0 + mt + m't' \\ z = z_0 + nt + n't' \end{cases} \qquad t,t' \in \mathbb{R} \qquad rk\begin{pmatrix} l & m & n \\ l' & m' & n' \end{pmatrix} = 2$$

$P_0 = (x_0, y_0, z_0) \in \alpha$

$V = \mathcal{L}(v_1, v_2)$ SPAZIO DI TRASLAZIONE

$v_1 = l e_1 + m e_2 + n e_3$

$v_2 = l' e_1 + m' e_2 + n' e_3$

<u>∞^2 punti</u>

EQ. CARTESIANA

$\alpha: ax + by + cz + d = 0 \qquad (a,b,c) \neq (0,0,0)$

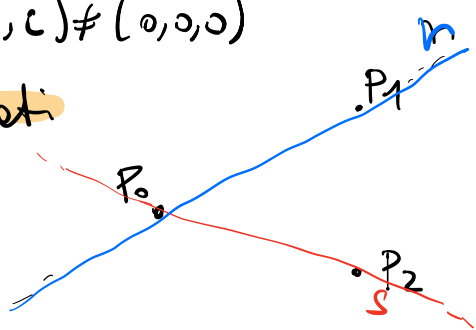
EQ. PIANO PER 3 punti non allineati

1)

$$\det\begin{pmatrix} x-x_0 & y-y_0 & z-z_0 \\ x_1-x_0 & y_1-y_0 & z_1-z_0 \\ x_2-x_0 & y_2-y_0 & z_2-z_0 \end{pmatrix} = 0$$

pd r

pd s



2) $\det \begin{pmatrix} x & y & z & 1 \\ x_0 & y_0 & z_0 & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \end{pmatrix} = 0$

CONDIZIONI DI PARALLELISMO

RETTA-RETTA in $A_2(\mathbb{R})$

$r: ax+by+c=0 \qquad s: a'x+b'y+c'=0$

$r \parallel s \iff \operatorname{rk}\begin{pmatrix} a & b \\ a' & b' \end{pmatrix} = 1 \iff \operatorname{rk}\begin{pmatrix} -b & a \\ -b' & a' \end{pmatrix} = 1$

RETTA-RETTA in $A_3(\mathbb{R})$

$r: \begin{cases} ax+by+cz+d=0 \\ a'x+b'y+c'z+d'=0 \end{cases} \qquad s: \begin{cases} a''x+b''y+c''z+d''=0 \\ a'''x+b'''y+c'''z+d'''=0 \end{cases}$

$\operatorname{rk}\begin{pmatrix} a & b & c \\ a' & b' & c' \end{pmatrix} = 2 \qquad \operatorname{rk}\begin{pmatrix} a'' & b'' & c'' \\ a''' & b''' & c''' \end{pmatrix} = 2$

$\text{pdr} = [(l, m, n)] \qquad \text{pds} = [(l', m', n')]$

$r \parallel s \iff \operatorname{rk}\begin{pmatrix} l & m & n \\ l' & m' & n' \end{pmatrix} = 1 \iff \operatorname{rk}\begin{pmatrix} a & b & c \\ a' & b' & c' \\ a'' & b'' & c'' \\ a''' & b''' & c''' \end{pmatrix} = 2$

RETTA-PIANO in $A_3(\mathbb{R})$

$\pi: ax+by+cz+d=0$

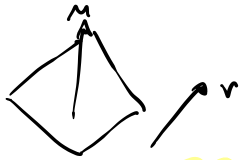
$n = (a, b, c)$

$\text{giacitura} = [(l', m', n'), (l'', m'', n'')]$

$r: \begin{cases} a'x+b'y+c'z+d'=0 \\ a''x+b''y+c''z+d''=0 \end{cases}$

$\operatorname{rk}\begin{pmatrix} a' & b' & c' \\ a'' & b'' & c'' \end{pmatrix} = 2 \qquad \text{pdr} = [(l, m, n)]$

$$\text{rk}\begin{pmatrix} \ell' & m' & n' \\ \ell'' & m'' & n'' \end{pmatrix} = 2$$



$$\pi \parallel r \iff \text{rk}\begin{pmatrix} \ell & m & n \\ \ell' & m' & n' \\ \ell'' & m'' & n'' \end{pmatrix} = 2 \iff (a,b,c)\cdot(\ell,m,n) = a\ell + bm + cn = 0$$

ESEMPIO

Eq. del piano passante per $A=(0,1,1)$, $B=(1,1,0)$ e $C=(0,0,1)$

1) $$\det\begin{pmatrix} x-0 & y-1 & z-1 \\ 1-0 & 1-1 & 0-1 \\ 0-0 & 0-1 & 1-1 \end{pmatrix} = \det\begin{pmatrix} x & y-1 & z-1 \\ 1 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}$$

$$= -(-1)\det\begin{pmatrix} x & z-1 \\ 1 & -1 \end{pmatrix}$$

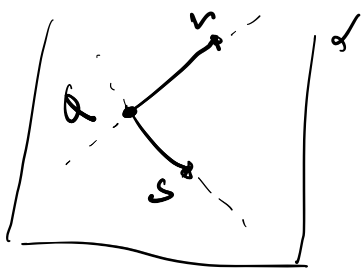
$$= -x - (z-1) = 0$$

α: $\boxed{x + z - 1 = 0}$

ESEMPIO

Piano passante per $Q=(1,0,2)$ e giacitura $[(1,-1,0),(0,1,2)]$

p.d. delle due rette $\subseteq$ nel piano



$\boxed{2x + 2y - z = 0}$

$$\alpha: \det\begin{pmatrix} x-1 & y-0 & z-2 \\ 1 & -1 & 0 \\ 0 & 1 & 2 \end{pmatrix} = 0 \Rightarrow -2(x-1) + z - 2 - 2y = 0$$

$$-2x + \cancel{2} + z - \cancel{2} - 2y = 0$$

ESERCIZIO

Determinare piano $P=(3,1,-1)$ e parallelo alle rette:

$$r: \begin{cases} x-y+z=2 \\ y+z=0 \end{cases} \qquad s: \begin{cases} x=1+2t \\ y=3t \\ z=-t \end{cases} \quad t\in\mathbb{R}$$

N.B.

Se $r \parallel s$ allora $\exists\ \infty$ piani che contengono r e s. Se $r \nparallel s$ $\exists!$ piano.

$$r: \begin{cases} x-t-t=2 \\ y=t \\ z=-t \end{cases} \Rightarrow \begin{cases} x=2t+2 \\ y=t \\ z=-t \end{cases}$$

$pdr=[(2,1,-1)]$ $\qquad$ $pds=[(2,3,-1)]$

Il piano α cercato contiene una retta $\parallel r$ ed una $\parallel s \Rightarrow$ giacitura del piano è
$[(2,1,-1),(2,3,-1)]$

$$\det\begin{pmatrix} x-3 & y-1 & z+1 \\ 2 & 1 & -1 \\ 2 & 3 & -1 \end{pmatrix}=0$$

$$-(x-3)-2\cancel{(y-1)}+6(z+1)-2(z+1)+3(x-3)+2\cancel{(y-1)}=0$$

$$2(x-3)+4(z+1)=0 \Rightarrow x-3+2z+2=0$$

$$\boxed{x+2z-1=0}$$

FASCI DI PIANI IN $A_3(\mathbb{R})$

Def. Si dice FASCIO IMPROPRIO l'insieme di tutti i piani di $A_3(\mathbb{R})$ paralleli ad un piano π_0 dato.

$\pi_0 : ax+by+cz+d=0$

$\mathcal{F} : ax+by+cz+k=0 \quad k\in\mathbb{R} \Rightarrow \infty^1$ piani

Def. Si dice FASCIO PROPRIO l'insieme di tutti e soli i piani di $A_3(\mathbb{R})$ passanti una retta r_0 data.

Siano $\pi : ax+by+cz+d=0$ e $\pi' : a'x+b'y+c'z+d'=0$ due piani distinti passanti per r_0 ($\pi \not\parallel \pi'$).

$$\mathcal{F}_{r_0} : \alpha(ax+by+cz+d)+\beta(a'x+b'y+c'z+d')=0$$

$$(\alpha,\beta)\in\mathbb{R}^2\setminus\{(0,0)\}$$

∞^1 piani $\quad r_0$ è detta ASSE o SOSTEGNO del fascio.

ESERCIZIO Determinare il piano π passante per $A=(3,-1,0)$ e contenente

$$r: \begin{cases} x-y=1 \\ 2x+z=-1 \end{cases}$$

$$F_r: \ \alpha(x-y-1)+\beta(2x+z+1)=0$$

$$(\alpha,\beta)\neq(0,0)$$

impongo il passaggio per $A=(3,-1,0)$

$$\alpha(3+\cancel{1}-\cancel{1})+\beta(6+1)=0$$

$$3\alpha+7\beta=0 \Rightarrow \alpha=-\frac{7}{3}\beta$$

$$\beta\neq0$$

$$-\frac{7}{3}\cancel{\beta}(x-y-1)+\cancel{\beta}(2x+z+1)=0$$

$$7x-7y-7-6x-3z-3=0$$

$$\Downarrow$$

$$\pi: \ \boxed{x-7y-3z-10=0}$$

ESERCIZIO

Piano α passante per $A=(3,0,1)$ e $//$ al piano $\pi: 2x+y-z=5$

$$\mathcal{F}_\pi: 2x+y-z+k=0 \qquad k\in\mathbb{R}$$

Passaggio per $A=(3,0,1)$

$$6+0-1+k=0 \Rightarrow k=-5$$

$$\boxed{\alpha: 2x+y-z-5=0}$$

$\alpha=\pi$ perché $A\in\pi$

ESERCIZIO

Determinare, se esiste, il piano contenente le rette

$$r: \begin{cases} x+y-z=0 \\ x+z=1 \end{cases} \qquad s: \begin{cases} 2x+y=0 \\ y-2z=3 \end{cases}$$

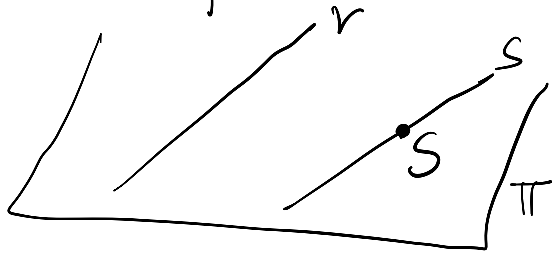
$$A|B=\left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 1 & 0 & 1 & 1 \\ 2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 3 \end{array}\right)$$

$\uparrow C_1-2C_2$

$$\det(A|B)=0$$

$\Rightarrow r$ e s complanari $\Rightarrow \exists\, \pi \supseteq r,s$

Prendo un fascio di piani $\mathcal{F}_r$ di supporto r e scelgo un punto $S \in s$ e $S \notin r$



$$\mathcal{F}_r : \quad \alpha(x+y-z)+\beta(x+z-1)=0$$

$$S=\left(0,0,-\frac{3}{2}\right) \in s \ , \notin r$$

Impongo il passaggio per S

$$\alpha\left(\frac{3}{2}\right)+\beta\Big(\underbrace{-\frac{3}{2}-1}_{-\frac{5}{2}}\Big)=0$$

$$3\alpha-5\beta=0 \Rightarrow \alpha=\frac{5}{3}\beta \ , \ \beta\neq 0$$

$$\frac{5}{3}\cancel{\beta}(x+y-z)+\cancel{\beta}(x+z-1)=0$$

$$5x+5y-5z+3x+3z-3=0$$

$$\boxed{8x+5y-2z-3=0}$$