$$1 + \rho\begin{pmatrix} 0 & 0 & -2 \\ -1 & -3 & 4 \end{pmatrix} = \boxed{3}$$

$$\rho(A|B) = \begin{cases} 4 & h \neq 0,-2 \\ 3 & h = -2 \\ 2 & h = 0 \end{cases}$$

$$\boxed{h=0} \Rightarrow \rho(A) = \rho(A|B)$$

INTERSEZIONE E SOMMA DI SOTTOSPAZI

Siano $U, W \leq V(\mathbb{K})$ $\quad U \cap W \leq V(\mathbb{K})$

$U \cup W \nleq V(\mathbb{K})$

Def.

Dati U e $W \leq V(\mathbb{K})$ si dice **somma** di U e W:

$$U+W = \{v \in U+w \mid v \in U \text{ e } w \in W\} \leq V(\mathbb{K})$$

Il sottospazio $U+W$ è il più piccolo sottospazio che contiene $U \cup W$!

LA SOMMA SI DICE DIRETTA $\Leftrightarrow$ $U \cap W = \{\underline{0}\}$ vettore nullo el. neutro di $(V,+)$

$U+W = V$

FORMULA DI GRASSMAN $\quad U, W \leq V(\mathbb{K})$

$$\dim(U+W) = \dim(U) + \dim(W) - \dim(U \cap W)$$

U e W SONO IN SOMMA DIRETTA $\Leftrightarrow$ $\dim(U+W) = \dim(U) + \dim(W) = \dim(V)$

ESERCIZIO

Dati $U = \left\{ \begin{pmatrix} x & y \\ z & t \end{pmatrix} \in M_2(\mathbb{R}) \mid y = x+z \right\}$

$W = \left\{ \begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix} \mid \alpha, \beta \in \mathbb{R} \right\}$

determinare $U \cap W$, $U + W$ e le loro dimensioni.

$$U = \left\{ \begin{pmatrix} x & x+z \\ z & t \end{pmatrix} \mid x,z,t \in \mathbb{R} \right\} = \left\{ x\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} + z\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + t\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

$$= \mathcal{L}\left(\left\{ \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\} \right)$$

$$\dim(U) = \operatorname{rk}\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 1 + \operatorname{rk}\begin{pmatrix} 1 & 0 \\ \cancel{1 \quad 1} \\ 0 & 1 \end{pmatrix} = 1 + \operatorname{rk}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 3$$

$$W = \left\{ \alpha\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \beta\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\} = \mathcal{L}\left(\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\} \right)$$

$$\dim(W) = \operatorname{rk}\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} = \operatorname{rk}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 2$$

$U \cap W$

$$\begin{cases} x = \alpha \\ x+z = \beta \\ z = \beta \\ t = \alpha \end{cases} \Rightarrow \begin{cases} \alpha = 0 \\ x = 0 \\ t = \beta \\ t = 0 \end{cases}$$

$$U \cap W = \left\{ \begin{pmatrix} 0 & \beta \\ \beta & 0 \end{pmatrix} \mid \alpha, \beta \in \mathbb{R} \right\}$$

$$= \mathcal{L}\left(\left\{ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\} \right)$$

$$\dim(U \cap W) = 1$$

$$B_{U \cap W} = \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right)$$

$$B_U = \left(\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right)$$

$$B_W = \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right)$$

$B_U \cup B_W$ È SIST. GENERATORI PER $U+W$ QUINDI

$$\dim(U+W) = rk \begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix} = rk \begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

$C_5 = C_2$

$$= 1 + rk \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} = 2 + rk \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} = \boxed{4}$$

$$B_{U+W} = \left(\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right)$$

$$U+W = \left\{ \begin{pmatrix} x+t & x+y \\ y & z+t \end{pmatrix} \mid x,y,z,t \in \mathbb{R} \right\}$$

$$\dim(U+W) = \underset{3}{\dim(U)} + \underset{2}{\dim(W)} - \underset{1}{\dim(U \cap W)} = 4 \quad \text{V}$$

ESERCIZIO

Dati $U = \{(a,b,b,0) \mid a,b \in \mathbb{R}\}$

$W = \{(x,y,z,t) \mid y = 2x+2z, t=0\}$

1) $\dim U$, $\dim W$
2) $U \cap W$, $U+W$ e le loro dimensioni

$U = \mathcal{L}(\{(1,0,0,0),\ (0,1,1,0)\})$

$B_U = ((1\ 0\ 0\ 0),\ (0\ 1\ 1\ 0))$

$\dim U = |B_U| = 2$

$W = \{(x, 2x+2z, z, 0)\} = \mathcal{L}(\{(1,2,0,0), (0,2,1,0)\})$

$B_W = ((1,2,0,0), (0,2,1,0))$

$\dim W = 2$

$U \cap W = \{(x, 2x+2z, z, 0) \mid x = a,\ 2x+2z = b,\ z = b\}$

$\begin{cases} 2x = -b & z = b \\ 2x = 2a \end{cases}$

$2a = -b$

$= \{(-\frac{b}{2}, b, b, 0) \mid b \in \mathbb{R}\}$

$= \mathcal{L}(\{(-1,2,2,0)\})$

$B_{U\cap W} = ((-1,2,2,0)) \qquad \dim U \cap W = 1$

$\dim(U+W) = 2+2-1 = 3$

$= \text{rk}\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} = 1 + \text{rk}\begin{pmatrix} 1 & 2 & 2 \\ 1 & 0 & 1 \\ \cancel{0} & \cancel{0} & \cancel{0} \end{pmatrix} = \boxed{3}$

$B_{U+W} = ((1,0,0,0), (0,1,1,0), (1,2,0,0))$

$U+W = \{ (x+z, y+2t, t, 0) \mid x,y,z \in \mathbb{R} \}$

ESERCIZIO

Dati $U = \mathcal{L}(\{(1,0,h,0), (1,0,1,0)\})$

$W = \{(x,y,z,t) \mid x+y+z=0\}$

$W = \{(x,y,-x-y,t)\} = \mathcal{L}(\{(1,0,-1,0), (0,1,-1,0), (0,0,0,1)\})$

Per quali $h \in \mathbb{R}$ $U \oplus W$?

$\dim(U) = \begin{cases} 2 & h \neq 1 \\ 1 & h=1 \end{cases}$ $\dim(W) = 3$ $\qquad U+W \leq \mathbb{R}^4$ $\underline{\dim(U+W) \leq 4}$

SIGNIFICA CHE $B_U \cup B_W$ È BASE PER $\underline{U+W}$

$h \neq 1$ NON SONO IN SOMMA DIRETTA

$h=1$

$$\operatorname{rk}\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = 1 + \operatorname{rk}\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & -1 \end{pmatrix}$$

$$= 2 + \operatorname{rk}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \boxed{4} \quad ✓$$

$\boxed{h=1}$ $\underline{\text{SOMMA DIRETTA}}$

ESERCIZIO

In $\mathbb{R}^{6,3}$, determinare le possibili dimensioni di $U \cap W$ dove $\dim(U)=9$ $\dim(W)=12$.

$\mathbb{R}^{6,3} \simeq \mathbb{R}^{18}$

$\dim(\mathbb{R}^{6,3})=18$

$$\dim(U\cap W) = \dim(U)+\dim(W)-\dim(U+W)$$

$$\max(\underbrace{\dim(U)}_{9}, \underbrace{\dim(W)}_{12}) \leq \dim(U+W) \leq \min(\underbrace{\dim(V)}_{18}, \underbrace{\dim(U)+\dim(W)}_{21})$$

$$12 \leq \dim(U+W) \leq 18$$

"=" per $U \subseteq W$ "=" per $U+W=V=\mathbb{R}^{6,3}$

$$\max(0, \dim(U)+\dim(W)-\dim(V)) \leq \dim(U\cap W) \leq \min(\underbrace{\dim(U)}_{9}, \underbrace{\dim(W)}_{12})$$

$21-18=3$ "=" se $U \subseteq W$

$$3 \leq \dim(U\cap W) \leq 9$$

ESERCIZIO

In $\mathbb{R}^4$, al variare di $k \in \mathbb{R}$, si determini se

$U_k = \{(x,y,z,t) \mid x+y-z=0,\ x-ky=0\}$

$W_k = \mathcal{L}((1,1,k,0),(k,1,1,0))$

SONO IN SOMMA DIRETTA.

$$U_k = \{ (x,y,z,t) \mid x = ky,\ z = (k+1)y \}$$
$$= \{ (ky, y, (k+1)y, t) \mid y,t \in \mathbb{R} \} = \mathcal{L}((k,1,k+1,0), (0,0,0,1))$$

$$\dim(U_k) = rk \begin{pmatrix} k & 0 \\ \boxed{1 \quad 0} & \\ k+1 & 0 \\ \boxed{0 \quad 1} & \end{pmatrix} = 2$$

$$\dim(W_k) = rk \begin{pmatrix} 1 & k \\ 1 & 1 \\ k & 1 \\ 0 & 0 \end{pmatrix} = rk \begin{pmatrix} 1 & k \\ 1 & 1 \\ k & 1 \end{pmatrix} = rk \begin{pmatrix} 1 & k \\ 0 & 1-k \\ 0 & 1-k^2 \end{pmatrix}$$

$$= 1 + rk \begin{pmatrix} 1-k \\ \underbrace{1-k^2}_{(1-k)(1+k)} \end{pmatrix}$$

$$= \begin{cases} 2 & k \neq 1 \\ 1 & k = 1 \end{cases}$$

SOMMA DIRETTA se e solo se

$$\dim(U_k + W_k) \overset{*}{=} \dim(U_k) + \dim(W_k) \overset{*}{=} \dim(V)$$

solo se $B_{U_k} \cup B_{W_k}$ È BASE PER $U_k + W_k$

$\Rightarrow \dim(U_k \cap W_k) = 0$

solo se $U_k + W_k = V$

$k = 1 \Rightarrow$ NON È SOMMA DIRETTA PERCHÉ $2 + 1 \neq 4$

$k \neq 1$

$$\dim(U_k + W_k) = rk \begin{pmatrix} \overbrace{k \quad 0}^{B_{U_k}} & \overbrace{1 \quad k}^{B_{W_k}} \\ 1 \quad 0 & 1 \quad 1 \\ k+1 \quad 0 & k \quad 1 \\ 0 \quad 1 & 0 \quad 0 \end{pmatrix} = 1 + rk \begin{pmatrix} k & 1 & k \\ 1 & 1 & 1 \\ k+1 & k & 1 \end{pmatrix}$$

$$= \begin{cases} 4 & k \neq 0, 1 \\ 3 & k = 0, 1 \end{cases}$$

$$\det\begin{pmatrix} k & 1 & k \\ 1 & 1 & 1 \\ k+1 & k & 1 \end{pmatrix} = k + k + \cancel{1} + \cancel{k^2} - k(k+1) - \cancel{k^2} - \cancel{1} = 2k - k^2 - k = k - k^2$$
$$= k(1-k)$$

$\forall k \neq 0, 1$ SONO IN <u>SOMMA DIRETTA</u>

ESERCIZIO

Al variare di $k \in \mathbb{R}$, si determini la dimensione della intersezione di

$$U = \mathcal{L}((1,1,k,1),(0,1,1,0)) \quad W = \mathcal{L}((1,0,0,1),(1,0,1-k,0))$$

$$\dim U = \operatorname{rk}\begin{pmatrix} 1 & 0 \\ 1 & 1 \\ k & 1 \\ 1 & 0 \end{pmatrix} = 2 \qquad \dim W = \operatorname{rk}\begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 1-k \\ 1 & 0 \end{pmatrix} = 2$$

$$\dim(U+W) = \operatorname{rk}\begin{pmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ k & 1 & 0 & 1-k \\ 1 & 0 & 1 & 0 \end{pmatrix} = \operatorname{rk}\begin{pmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ k & 1 & 0 & 1-k \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$= 1 + \operatorname{rk}\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ k & 1 & 0 \end{pmatrix} = 2 + \operatorname{rk}\begin{pmatrix} 1 & 1 \\ k & 1 \end{pmatrix}$$

$$= 2 + \operatorname{rk}\begin{pmatrix} 1-k & 0 \\ k & 1 \end{pmatrix}$$

$$= 3 + \operatorname{rk}(1-k) = \begin{cases} 4 & k \neq 1 \\ 3 & k = 1 \end{cases}$$

<u>GRASSMANN</u>

$$\dim(U \cap W) = \dim(U) + \dim(W) - \dim(U+W)$$
$$= 4 - \dim(U+W) = \begin{cases} 1 & k = 1 \\ 0 & k \neq 1 \end{cases}$$

COMPLEMENTO DIRETTO

Def. Dato $U \leq V$, un COMPLEMENTO DIRETTO di U è un sottospazio $W \leq V$ tale che

$$\begin{cases} U + W = V \\ U \cap W = \{0\} \end{cases} \Rightarrow U \oplus W$$

ESERCIZIO

Si determini una BASE di un complemento diretto di

$U = \{(x, y, z) \mid x + y = 0\} \leq \mathbb{R}^3$

$U = \{(x, -x, z) \mid x, z \in \mathbb{R}\} = \mathcal{L}((1, -1, 0), (0, 0, 1))$

Devo trovare W tale che $\begin{cases} U + W = V \\ U \cap W = \{0\} \end{cases}$ cioè

$\dim(W) = 1$ (un solo elemento di base) e $\dim(U+W) = 3$

$$\underset{3}{\underset{\parallel}{\dim(U+W)}} = \underset{2}{\underset{\parallel}{\dim(U)}} + \underset{1}{\underset{\parallel}{\dim(W)}} - \underset{0}{\underset{\parallel}{\dim(U \cap W)}}$$

$$\dim(U+W) = rk \begin{array}{c} \begin{array}{ccc} B_U & & B_W \end{array} \\ \begin{pmatrix} 1 & 0 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \end{array}$$

COMPLETO LA BASE DI $U+W$ CON EL. DELLA <u>BASE CANONICA</u>.

$= 3$

ESERCIZIO

In $M_2(\mathbb{R})$ si determini un complemento diretto di

$$U = \left\{ \begin{pmatrix} 3a-b & b+c \\ 3a+c & 0 \end{pmatrix} \in M_2(\mathbb{R}) \;\middle|\; a,b,c \in \mathbb{R} \right\}$$

$$U = \left\{ a\begin{pmatrix} 3 & 0 \\ 3 & 0 \end{pmatrix} + b\begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}$$

$$= \mathcal{L}\left(\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right) = \mathcal{L}\left(\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right)$$

$$\dim(U) = \operatorname{rk}\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ \cancel{0} & \cancel{0} & \cancel{0} \end{pmatrix} = \operatorname{rk}\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} = \operatorname{rk}\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} = 1+1=2$$

Devo trovare un W tale che $\begin{cases} W \cap U = \{0\} \\ W + U = V = M_2(\mathbb{R}) \end{cases} \Rightarrow \begin{cases} \dim(U+W) = 4 \\ \dim(U) + \dim(W) = 4 \end{cases}$

$$\Downarrow$$

$$\dim(W) = 2$$

$$\dim(U+W) = \begin{array}{c} \begin{array}{cc} \overbrace{\quad\;\;}^{B_U} & \overbrace{\quad\;\;}^{B_W} \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array} = 2 + \operatorname{rk}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \boxed{4}$$

$$W = \mathcal{L}\left(\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right)$$

ESERCIZIO

Dato $W = \mathcal{L}\left(\left\{ \begin{pmatrix} a+1 & a+b+1 \\ 0 & b-1 \end{pmatrix} \;\middle|\; a,b \in \mathbb{R} \right\} \right) \leq M_2(\mathbb{R})$

$$= \mathcal{L}\left(\left\{ a\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} \right\} \right)$$

$$= \mathcal{L}\left(\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} \right)$$

$$S = \left\{ \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} \right\}$$

INSIEME DI GENERATORI

1) Base di W e $\dim W$

$$\dim W = \operatorname{rk} \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ \cancel{0} & \cancel{0} & \cancel{0} \\ 0 & 1 & -1 \end{pmatrix} = \operatorname{rk} \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & -1 \end{pmatrix} = \operatorname{rk} \begin{pmatrix} \cancel{1} & \cancel{0} & \cancel{1} \\ 0 & 1 & 0 \\ 0 & 1 & -1 \end{pmatrix}$$

$$= 1 + \operatorname{rk} \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} = 3 \Rightarrow \underline{S \text{ È LIBERO}}$$

$$B_W = \left(\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} \right) \qquad \dim W = |B_W| = 3$$

2) Le componenti di $M = \begin{pmatrix} 0 & 2 \\ 0 & 1 \end{pmatrix}$ rispetto a B_W

$$\alpha \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} + \beta \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} + \gamma \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} \alpha & \alpha \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & \beta \\ 0 & \beta \end{pmatrix} + \begin{pmatrix} \gamma & \gamma \\ 0 & -\gamma \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 0 & 1 \end{pmatrix}$$

$$\begin{cases} \alpha + \gamma = 0 & \Rightarrow \alpha = -1 \\ \alpha + \beta + \gamma = 2 & \Rightarrow \beta = 2 \\ \beta - \gamma = 1 & \Rightarrow \gamma = 1 \end{cases}$$

COMPONENTI DI M RISPETTO A B

$$\phi_{B_W}(M) = (-1, 2, 1)$$