AMIR Exercises 2025/26

Vector Spaces

Hilbert Spaces

Text and Exercices are from the textbook

Foundations of Signal Processing

By M. Vetterli, J. Kovacevic and V.K. Goyal

Vector spaces

DEFINITION 2.5 (LINEARLY INDEPENDENT SET) The set of vectors $\{\varphi_0, \varphi_1, \ldots, \varphi_{N-1}\}$ is called *linearly independent* when $\sum_{k=0}^{N-1} \alpha_k \varphi_k = \mathbf{0}$ is true only if $\alpha_k = 0$ for all k. Otherwise, the set is linearly dependent. An infinite set of vectors is called linearly independent when every finite subset is linearly independent.

DEFINITION 2.7 (INNER PRODUCT) An *inner product* on a vector space V over $\mathbb{C}$ (or $\mathbb{R}$) is a complex-valued (or real-valued) function $\langle \cdot, \cdot \rangle$ defined on $V \times V$ with the following properties for any $x, y, z \in V$ and $\alpha \in \mathbb{C}$ (or $\mathbb{R}$):

(i) *Distributivity:* $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$.

(ii) *Linearity in the first argument:* $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$.

(iii) *Hermitian symmetry:* $\langle x, y \rangle^* = \langle y, x \rangle$.

(iv) *Positive definiteness:* $\langle x, x \rangle \geq 0$, and $\langle x, x \rangle = 0$ if and only if $x = \mathbf{0}$.

Exercise

2.5. *Linear independence*
Find the values of the parameter $a \in \mathbb{C}$ such that the following set is linearly independent:

$$U = \left\{ \begin{bmatrix} 0 & a^2 \\ 0 & j \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & a-1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ ja & 1 \end{bmatrix} \right\}.$$

For $a = j$, express the matrix $\begin{bmatrix} 0 & 5 \\ 2 & j-2 \end{bmatrix}$ as a linear combination of elements of U.

Definition 2.5 (Linearly independent set) The set of vectors $\{\varphi_0, \varphi_1, \ldots, \varphi_{N-1}\}$ is called *linearly independent* when $\sum_{k=0}^{N-1} \alpha_k \varphi_k = \mathbf{0}$ is true only if $\alpha_k = 0$ for all k. Otherwise, the set is linearly dependent. An infinite set of vectors is called linearly independent when every finite subset is linearly independent.

Exercise 2.5

① For $U = \left\{ \begin{bmatrix} 0 & a^2 \\ 0 & j \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & a-1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ ja & 1 \end{bmatrix} \right\}$ to be an independent set, it is necessary and sufficient that

$$\lambda_0 \begin{bmatrix} 0 & a^2 \\ 0 & j \end{bmatrix} + \lambda_1 \begin{bmatrix} 0 & 1 \\ 1 & a-1 \end{bmatrix} + \lambda_2 \begin{bmatrix} 0 & 0 \\ ja & 1 \end{bmatrix} = 0$$

for λ_0, λ_1 and $\lambda_2 \in \mathbb{C}$, have a UNIQUE solution $\lambda_0 = \lambda_1 = \lambda_2 = 0 \rightarrow$ This equation is equivalent to the system

$$\begin{cases} a^2\lambda_0 + \lambda_1 = 0 \\ \lambda_1 + ja\lambda_2 = 0 \\ j\lambda_0 + (a-1)\lambda_1 + \lambda_2 = 0 \end{cases}$$

$$\Rightarrow a^2\lambda_0 = -\lambda_1 = ja\lambda_2 \quad (*)$$

Now by multiplying by a^2 the last eq. and substituting for λ_0 and λ_1

$$ja^2\lambda_0 + (a-1)a^2\lambda_1 + a^2\lambda_2 = j\left(j\frac{1}{a}\right)a^2\lambda_2 - (a-1)a^2(ja\lambda_2) + a^2\lambda_2 =$$

$$= \left(-a - j(a-1)a^3 + a^2\right)\lambda_2 = a\left(-1 - j(a-1)a^2 + a\right)\lambda_2 =$$

$$= a\left(1 - ja^2\right)(a-1)\lambda_2 = a(1-ay)(1+ay)(a-1)\lambda_2$$

where $\gamma = \sqrt{j} = (1+j)/\sqrt{2}$

Assuming $a \notin \left\{ 0, 1, -\frac{1}{\gamma}, \frac{1}{\gamma} \right\} \Rightarrow \lambda_2 = 0$

(and) $\Rightarrow$ also $\lambda_0 = \lambda_1 = 0$ (see (*))

$$\boxed{\begin{array}{l}\text{Hence } U \text{ is an independent set iff} \\ \mathbb{C} \ni a \notin \left\{ 0, 1, (1-j)/\sqrt{2}, -(1-j)/\sqrt{2} \right\}\end{array}}$$

n.b. $\frac{1}{\gamma} = \frac{\sqrt{2}}{1+j} \frac{1-j}{1-j} = \frac{\sqrt{2}}{2} 1-j = \frac{1-j}{\sqrt{2}}$

② For $a=j$ $\quad \lambda_0 \begin{bmatrix} 0 & a^2 \\ 0 & j \end{bmatrix} + \lambda_1 \begin{bmatrix} 0 & 1 \\ 1 & a-1 \end{bmatrix} + \lambda_2 \begin{bmatrix} 0 & 0 \\ j^2 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 5 \\ 2 & j-2 \end{bmatrix}$

$$\rightarrow \begin{cases} -\lambda_0 + \lambda_1 = 5 \\ \lambda_1 - \lambda_2 = 2 \\ j\lambda_0 + (j-1)\lambda_1 + \lambda_2 = j-2 \end{cases} \longrightarrow \begin{cases} \lambda_0 = \lambda_1 - 5 \\ \lambda_2 = \lambda_1 - 2 \rightarrow -\lambda_1 + \lambda_2 = -2 \\ j\lambda_0 + j\lambda_1 = j \rightarrow \lambda_1 = 1 - \lambda_0 \end{cases}$$

$$\rightarrow \begin{cases} \lambda_0 = 1 - \lambda_0 - 5 \rightarrow \lambda_0 = -2 \\ -2 + \lambda_1 = 1 \rightarrow \lambda_1 = 3 \\ \lambda_2 = 3 - 2 \rightarrow \lambda_2 = 1 \end{cases}$$

thus we verify

that $(-2)\begin{bmatrix} 0 & -1 \\ 0 & j \end{bmatrix} + 3\begin{bmatrix} 0 & 1 \\ 1 & j-1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 5 \\ 2 & j-2 \end{bmatrix}$

Exercise

2.7. *Inner product on* $\mathbb{C}^N$
Prove that $\langle x, y \rangle = y^* A x$ is a valid inner product on $\mathbb{C}^N$ if and only if A is a Hermitian, positive definite matrix.

DEFINITION 2.7 (INNER PRODUCT) An *inner product* on a vector space V over $\mathbb{C}$ (or $\mathbb{R}$) is a complex-valued (or real-valued) function $\langle \cdot, \cdot \rangle$ defined on $V \times V$ with the following properties for any $x, y, z \in V$ and $\alpha \in \mathbb{C}$ (or $\mathbb{R}$):

(i) *Distributivity:* $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$.

(ii) *Linearity in the first argument:* $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$.

(iii) *Hermitian symmetry:* $\langle x, y \rangle^* = \langle y, x \rangle$.

(iv) *Positive definiteness:* $\langle x, x \rangle \geq 0$, and $\langle x, x \rangle = 0$ if and only if $x = \mathbf{0}$.

Exercise 2.7

To show that $\langle x, y\rangle = y^* A x$ is a valid inner product on $\mathbb{C}^N$ iff A is Hermitian we verify the definition 2.7 (i) to (iv)

(i) distributivity $\langle x+y, z\rangle = z^* A(x+y) =$
$= z^* A x + z^* A y = \langle x, z\rangle + \langle y, z\rangle$ with no specific conditions on A (the same for (ii))

(iii) $\rightarrow x^* A^* y = (y^* A x)^* = \langle x, y\rangle^* \overset{(iii)}{=} \langle y, x\rangle = x^* A y$
$\Rightarrow A = A^*$ i.e. A must be Hermitian

(iv) $\langle x, x\rangle > 0 \quad \forall x \neq 0 \Rightarrow x^* A x > 0 \quad \forall x \neq 0$
which is precisely the definition of A being positive definite

DEFINITION 2.8 (ORTHOGONALITY)

(i) Vectors x and y are said to be *orthogonal* when $\langle x, y\rangle = 0$, written as $x \perp y$.

(ii) A set of vectors S is called *orthogonal* when $x \perp y$ for every x and y in S such that $x \neq y$.

(iii) A set of vectors S is called *orthonormal* when it is orthogonal and $\langle x, x\rangle = 1$ for every x in S.

(iv) A vector x is said to be *orthogonal* to a set of vectors S when $x \perp s$ for all $s \in S$, written as $x \perp S$.

(v) Two sets S_0 and S_1 are said to be *orthogonal* when every vector s_0 in S_0 is orthogonal to the set S_1, written as $S_0 \perp S_1$.

(vi) Given a subspace S of a vector space V, the *orthogonal complement* of S, denoted $S^{\perp}$, is the set $\{x \in V \mid x \perp S\}$.

Definition 2.9 (Norm) A *norm* on a vector space V over $\mathbb{C}$ (or $\mathbb{R}$) is a real-valued function $\|\cdot\|$ defined on V with the following properties for any $x, y \in V$ and $\alpha \in \mathbb{C}$ (or $\mathbb{R}$):

(i) *Positive definiteness:* $\|x\| \geq 0$, and $\|x\| = 0$ if and only if $x = \mathbf{0}$.

(ii) *Positive scalability:* $\|\alpha x\| = |\alpha| \, \|x\|$.

(iii) *Triangle inequality:* $\|x+y\| \leq \|x\| + \|y\|$, with equality if and only if $y = \alpha x$.

Standard inner product on $\mathbb{C}^{\mathbb{Z}}$ The standard inner product on the vector space of complex-valued sequences over $\mathbb{Z}$ is

$$\langle x, y \rangle = \sum_{n \in \mathbb{Z}} x_n \, y_n^* = y^* \, x, \tag{2.22b}$$

where we are taking the unusual step of using matrix product notation with an infinite row vector y^* and an infinite column vector x.

Standard norm on $\mathbb{C}^{\mathbb{Z}}$ The standard norm on $\mathbb{C}^{\mathbb{Z}}$, induced by the inner product (2.22b), is:

$$\|x\| = \sqrt{\langle x, x \rangle} = \left(\sum_{n \in \mathbb{Z}} |x_n|^2 \right)^{1/2}. \tag{2.26b}$$

Exercise

2.10. *Orthogonal transforms and ∞ norm*
Orthogonal transforms conserve the 2 norm, but not others, in general. Consider the ∞ norm (2.39b).

(i) Consider the set of real orthogonal transforms T_2 on $\mathbb{R}^2$, that is, plane rotations and rotoinversions (2.238). Give the best lower and upper bounds a_2 and b_2 so that

$$a_2 \;\leq\; \|T_2 x\|_\infty \;\leq\; b_2 \tag{P2.10-1}$$

holds for all orthogonal T_2 and all vectors x of unit 2 norm.

(ii) Extend (P2.10-1) by giving the best lower and upper bounds a_N and b_N for the general case of real orthogonal transforms T_N on $\mathbb{R}^N$ with $N \geq 2$.

$$\|x\|_\infty \;=\; \max(|x_0|, |x_1|, \ldots, |x_{N-1}|).$$

Exercise 2.10

Recall that $\|x\|_\infty = \max(|x_0|, |x_1|, \dots, |x_N|)$

(i) The set of roto(+inversion) transforms T_2 on $\mathbb{R}^2$ is

$$T_2 = \begin{bmatrix} \cos\theta & \mp\sin\theta \\ \sin\theta & \pm\cos\theta \end{bmatrix}$$

rotation

rot + coord. inversion

is an orthogonal transform (i.e. preserves lengths and angles between vectors)

Given x with $\|x\|_2 = 1$ we can find the bounds a_2 and b_2 s.t. $a_2 \le \|T_2 x\|_\infty \le b_2$ $\forall T_2$ by considering what happens on the unit circle:

- b_2 is clearly 1 (max reachable value)
- a_2 is when both vector components are equal $\Rightarrow$ we found an orthogonal transform that does not conserve the $\|\cdot\|_\infty$ norm

$$a_2 = \frac{1}{\sqrt{2}}$$

(ii) now we consider T_N on $\mathbb{R}^N$ and we want to find the lower and upper bounds a_N and b_N respectively

Since again $\|x\|_2 = 1$, $x \in \mathbb{R}^N$, the upper bound on the unit hypersphere is $b_N = 1$.

The lower bound is again achieved when all the vector components are equal $\Rightarrow$ the diagonal of an hypercube with vertices on the unit hypersphere $\longrightarrow$ $a_N = \dfrac{1}{\sqrt{N}}$

$$\Rightarrow \quad \frac{1}{\sqrt{N}} \leq \|T_N x\|_\infty \leq 1$$

Properties of norms induced by an inner product

The following facts hold in any inner product space.

Pythagorean theorem This theorem generalizes a well-known fact from Euclidean geometry to any inner product space. The statement learned in elementary school involves the sides of a right triangle. In its more general form the theorem states that:

$$x \perp y \qquad \text{implies} \qquad \|x+y\|^2 \;=\; \|x\|^2 + \|y\|^2. \tag{2.27a}$$

$$\{x_k\}_{k\in\mathcal{K}} \text{ orthogonal} \qquad \text{implies} \qquad \Big\|\sum_{k\in\mathcal{K}} x_k\Big\|^2 \;=\; \sum_{k\in\mathcal{K}} \|x_k\|^2. \tag{2.27b}$$

Parallelogram law The parallelogram law of Euclidean geometry generalizes the Pythagorean theorem, and it too can be generalized to any inner product space. It states that:

$$\|x+y\|^2 + \|x-y\|^2 \;=\; 2(\|x\|^2 + \|y\|^2). \tag{2.28}$$

Cauchy–Schwarz inequality This widely used inequality states that:

$$|\langle x,\, y\rangle| \;\leq\; \|x\|\,\|y\|, \tag{2.29}$$

with equality if and only if $x = \alpha y$ for some scalar α.

$$\boxed{\cos\theta \;=\; \frac{\langle x,\, y\rangle}{\|x\|\,\|y\|}.}$$

Exercises

2.11. *Cauchy–Schwarz inequality, triangle inequality, and parallelogram law*
Prove the following:

(i) Cauchy–Schwarz inequality given in (2.29).

(ii) Triangle inequality given in Definition 2.9.

(iii) Parallelogram law given in (2.28).

(iv) In a normed vector space over the scalars $\mathbb{R}$, the inner product given by the *real polarization identity*

$$\langle x, y \rangle = \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2\right) \qquad \text{(P2.11-1)}$$

satisfies the distributivity, Hermitian symmetry, and positive definiteness properties. (More difficult to prove is the linearity in the first argument property, which also holds; together, these verify that (P2.11-1) is a valid inner product. Similarly, for a normed vector space over the scalars $\mathbb{C}$, by the *complex polarization identity*

$$\langle x, y \rangle = \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2 + j\|x+jy\|^2 - j\|x-jy\|^2\right) \qquad \text{(P2.11-2)}$$

gives a valid inner product.)

Exercise 2.11 (ii), (iii) and (iv)

(ii) proof of the "Triangle Inequality"

$$\|x+y\|^2 = \langle x+y, x+y\rangle = \|x\|^2 + \underbrace{\langle x,y\rangle}_{\leq \|x\|\|y\|} + \underbrace{\langle y,x\rangle}_{\leq \|y\|\|x\|} + \|y\|^2$$

Cauchy-Schwarz

$$\leq \|x\|^2 + 2\|x\|\|y\| + \|y\|^2 = (\|x\| + \|y\|)^2$$

where equality holds iff x is a scalar multiple of y (i.e. x and y are aligned vectors). Taking the square root

$$\|x+y\| \leq \|x\| + \|y\| \quad \text{c.v.d.}$$

(iii) proof of the "Parallelogram law"

$$\|x+y\|^2 + \|x-y\|^2 = \langle x+y, x+y\rangle + \langle x-y, x-y\rangle =$$

$$= \|x\|^2 + \|y\|^2 + \langle x,y\rangle + \langle y,x\rangle + \|x\|^2 + \|y\|^2 - \langle x,y\rangle - \langle y,x\rangle =$$

$$= 2(\|x\|^2 + \|y\|^2) \quad \text{c.v.d.}$$

(iv) polarization identity inner product

$$\langle x,y\rangle = \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2\right)$$

I. Distributivity

add and subtract

$$\langle x+y, z\rangle \overset{\text{polariz. id. def.}}{=} \frac{1}{4}\left(\|x+y+z\|^2 - \|x+y-z\|^2 \pm \|x+y+z\|^2\right) =$$

$$= \frac{1}{2}\|x+y+z\|^2 - \frac{1}{4}\left(\|x+y+z\|^2 + \|x+y-z\|^2\right) =$$

parallelogram law

$$= \frac{1}{2}\|x+y+z\|^2 - \frac{1}{2}\left(\|x+y\|^2 + \|z\|^2\right) =$$

parallelogram law with $(x+y+z)+z = (x+y+2z)$

$$= \frac{1}{4}\left(\|x+y+2z\|^2 + \|x+y\|^2\right) - \frac{1}{2}\|z\|^2 - \frac{1}{2}\left(\|x+y\|^2 + \|z\|^2\right) =$$

parallelogram law with $(x+z)+(y+z) = (x+y+2z)$

$$= \frac{1}{4}\|x+y+2z\|^2 - \frac{1}{4}\|x+y\|^2 - \|z\|^2 =$$

p. law to III and IV summands

$$= \frac{1}{2}\|x+z\|^2 + \frac{1}{2}\|y+z\|^2 - \frac{1}{4}\|x-y\|^2 - \frac{1}{4}\|x+y\|^2 - \|z\|^2 =$$

$$= \frac{1}{2}\|x+z\|^2 + \frac{1}{2}\|y+z\|^2 - \frac{1}{2}\|x\|^2 - \frac{1}{2}\|y\|^2 - \|z\|^2 =$$

$$= \frac{1}{2}\|x+z\|^2 - \frac{1}{2}\left(\|x\|^2 + \|z\|^2\right) + \frac{1}{2}\|y+z\|^2 - \frac{1}{2}\left(\|y\|^2 + \|z\|^2\right) =$$

p. law to the terms in ( )

$$= \frac{1}{4}\left(\|x+z\|^2 - \|x-z\|^2\right) + \frac{1}{4}\left(\|y+z\|^2 - \|y-z\|^2\right) =$$

by the polarization identity

$$= \langle x, z\rangle + \langle y, z\rangle$$

II. Hermitian symmetry: for the real polariz. id. x and y commute
$\Rightarrow \langle y, x\rangle = \langle x, y\rangle$

III. Positive definiteness: $\langle x, x\rangle = \frac{1}{4}\left(\|x+x\|^2 + \|x-x\|^2\right)$
$= \frac{1}{4}\left(\|2x\|^2 - 0\right) = \|x\|^2 \Rightarrow$ pos. def of the inner product follows from pos. def. of the norm.

DEFINITION 2.10 (METRIC, OR DISTANCE) In a normed vector space, the *metric*, or *distance* between vectors x and y is the norm of their difference:

$$d(x, y) = \|x - y\|.$$

A *distance*, or *metric* $d : V \times V \to \mathbb{R}$ is a function with the following properties:

(i) *Nonnegativity:* $d(x, y) \geq 0$ for every x, y in V.
(ii) *Symmetry:* $d(x, y) = d(y, x)$ for every x, y in V.
(iii) *Triangle inequality:* $d(x, y) + d(y, z) \geq d(x, z)$ for every x, y, z in V.
(iv) *Identity of Indiscernibles:* $d(x, x) = 0$ and $d(x, y) = 0$ implies $x = y$.

$\ell^2(\mathbb{Z})$: **Space of square-summable sequences** This is the normed vector space of square-summable complex-valued sequences, and it uses the inner product (2.22b) and the norm (2.26b):

$$\langle x, y \rangle = \sum_{n \in \mathbb{Z}} x_n y_n^*, \qquad \|x\| = \left(\sum_{n \in \mathbb{Z}} |x_n|^2 \right)^{1/2}. \tag{2.32}$$

This space is often referred to as the space of *finite-energy sequences*.

Exercises

2.13. *Distances not necessarily induced by norms*
A *distance*, or *metric* $d : V \times V \to \mathbb{R}$ is a function with the following properties:

(i) *Nonnegativity:* $d(x, y) \geq 0$ for every x, y in V.

(ii) *Symmetry:* $d(x, y) = d(y, x)$ for every x, y in V.

(iii) *Triangle inequality:* $d(x, y) + d(y, z) \geq d(x, z)$ for every x, y, z in V.

(iv) *Identity of Indiscernibles:* $d(x, x) = 0$ and $d(x, y) = 0$ implies $x = y$.

The *discrete metric* is given by

$$d(x, y) = \begin{cases} 0, & \text{if } x = y; \\ 1, & \text{if } x \neq y. \end{cases}$$

Show that the discrete metric is a valid distance and is not induced by any norm.

Exercise 2.13

To show that $d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$

(i) non negativity: $\forall x, y \in V$, $d(x,y) \geq 0$

(ii) symmetry: $\forall x, y \in V$ $\begin{cases} \text{if } x = y & d(x,y) = d(y,x) = 0 \\ \text{if } x \neq y & d(x,y) = d(y,x) = 1 \end{cases}$

(iii) Triangle inequality: $\forall x, y, z$ that are not ALL equal

$d(x,y) + d(y,z) \geq 1 \geq d(x,z)$, where

for $x = y = z$ $\qquad d(x,y) + d(y,z) = 0 = d(x,z)$

(iv) Identity of indiscernable $\forall x, y \in V$ by def $d(x,x) = 0$

and $d(x,y) = 0 \Rightarrow x = y$

To show that $d(x,y)$ is NOT bounded by any norm it is possible to give an example:

Consider $x = 2e_0$ $\quad \left[\cdots\ 0\ \ 0\ \ 0\ \ \underset{\underset{n=0}{\uparrow}}{2}\ \ 0\ \ 0\ \cdots\right]$

$y = -2e_1$ $\quad \left[\cdots\ 0\ \ 0\ \ 0\ \ 0\ \ \underset{\underset{n=1}{\uparrow}}{-2}\ \ 0\ \cdots\right]$

Then, for any $p \geq 1$

$$\|x-y\|_p = \left(2^p + 2^p\right)^{1/p} = 2^{(p+1)/p} \geq 2 > 1$$

while $d(x,y) = 1$

Standard normed vector spaces

$\mathbb{C}^N$ spaces As we said earlier, we can define other norms on $\mathbb{C}^N$. For example, the p *norm* is defined as

$$\|x\|_p = \left(\sum_{n=0}^{N-1} |x_n|^p\right)^{1/p}, \tag{2.39a}$$

for $p \in [1, \infty)$. Since the sum above has a finite number of terms, there is no doubt that the sums converge. Thus, we take as a vector space of interest the entire $\mathbb{C}^N$; note how this contrasts with some of the examples we see shortly ($\ell^p(\mathbb{Z})$ spaces).

For $p = 1$, this norm is called the *taxicab norm* or *Manhattan norm* because $\|x\|_1$ represents the driving distance from the origin to x following a rectilinear street grid. For $p = 2$, we get our usual Euclidean square norm from (2.39a), and only in that case is a p norm induced by an inner product. The natural extension of (2.39a) to $p = \infty$ (see Exercise 2.15) defines the ∞ norm as:

$$\|x\|_\infty = \max(|x_0|, |x_1|, \ldots, |x_{N-1}|). \tag{2.39b}$$

Using (2.39a) for $p \in (0, 1)$ does not give a norm but can still be a useful quantity. The failure to satisfy the requirements of a norm and an interpretation of (2.39a) with $p \to 0$ are explored in Exercise 2.16.

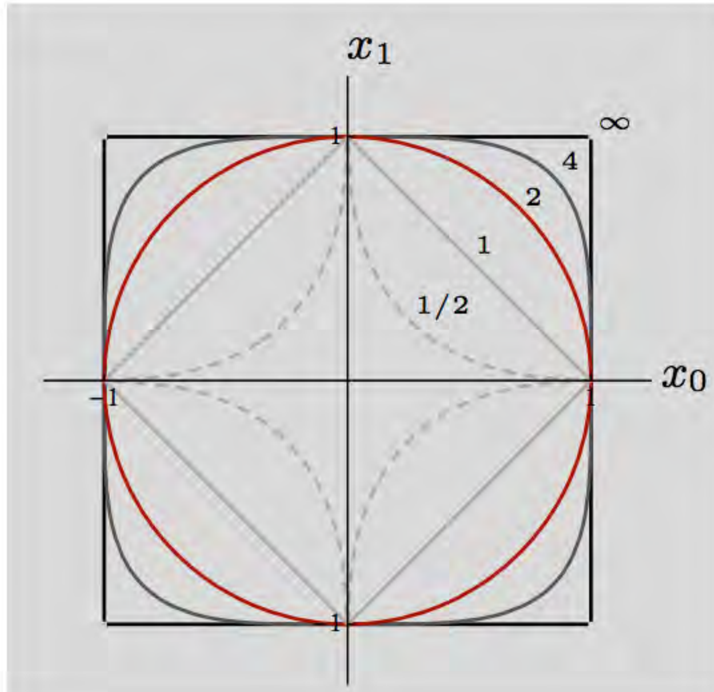


Figure 2.7 Sets of unit-norm vectors for different p norms: $p = \infty$, $p = 4$, $p = 2$, and $p = 1$ (from darkest to lightest), as well as for $p = 1/2$ (dashed), which is not a norm. Vectors ending on the curves are of unit norm in the corresponding p norm.

Exercises

2.15. *Definition of ∞ norm*
Show that the ∞ norm in (2.39b) is the natural extension of the p norm in (2.39a), by proving

$$\lim_{p\to\infty} \|x\|_p = \max_{i=0,\,1,\,\ldots,\,N-1} |x_i| \qquad \text{for any } x \in \mathbb{R}^N.$$

(*Hint:* Normalize x by dividing it by the entry of the largest magnitude. Compute the limit for the resulting vector.)

Exercise 2.15

Following the hint, we only need to consider

$$x = [1 \ a_1 \ a_2 \ a_3 \ \cdots\cdots \ a_{N-1}] \quad \text{with } |a_i| \leq 1$$

since any norm we consider satisfies $\|\alpha x\| = |\alpha| \|x\|$ (see def 2.9 (iii)) and since changing the order of elements does not influence the norm, conclusions we derive for x can be extended to any $y \in V$, and we need to show that $\lim_{p \to \infty} \|x\|_p = 1$. We try to found upper and lower bounds

a) because of the first element $\|x\|_p \geq 1$

b) we have $\|x\|_p^p = 1 + a_1^p + a_2^p + \cdots + a_{N-1}^p \leq N$ since $|a_i| \leq 1 \ \forall i$

$\Rightarrow \lim_{p \to \infty} \|x\|_p \leq \lim_{p \to \infty} N^{1/p} = 1$ $\Rightarrow$ combining a) and b) complete the proof

Exercises

2.16. *Quasinorms with $p < 1$*
Equation (2.39a) does not yield a valid norm when $p < 1$.

(i) Show that Definition 2.9(iii) fails to hold for (2.39a) with $p = 1/2$.
(ii) Show that for $x \in \mathbb{R}^N$, $\lim_{p\to 0} \|x\|_p^p$ gives the number of nonzero components in x.

$$\|x\|_p = \left(\sum_{n=0}^{N-1} |x_n|^p \right)^{1/p}$$

$$\|x+y\| \leq \|x\| + \|y\|$$ Triangle Inequality

Exercise 2.16

(i) let $x = [1 \ 0]^T$ and $y = [0 \ 1]^T$

Then $\|x+y\|_{1/2} = \left(1^{\frac{1}{2}} + 1^{\frac{1}{2}}\right)^2 =$

$= 4 > 2 = 1 + 1 = \|x\|_{1/2} + \|y\|_{1/2}$

↳ violating Triangle inequality

(ii) $\lim_{p\to 0} \|x\|_p^p = \lim_{p\to 0} \sum_{i=1}^{N} |x_i|^p = \sum_{i=1}^{N} \lim_{p\to 0} |x_i|^p$

$x \in \mathbb{R}^N$ $\Rightarrow$ $x_i \neq 0$ contributes 1

$x_i = 0$ " " 0

$\Rightarrow \lim_{p\to 0} \|x\|_p^p$ gives the count of non zero elements

Hilbert spaces

DEFINITION 2.13 (CONVERGENT SEQUENCE OF VECTORS) A sequence of vectors $x_0, x_1, \ldots$ in a normed vector space V is said to *converge* to $v \in V$ when $\lim_{k\to\infty} \|v - x_k\| = 0$. In other words: Given any $\varepsilon > 0$, there exists a K_ε such that

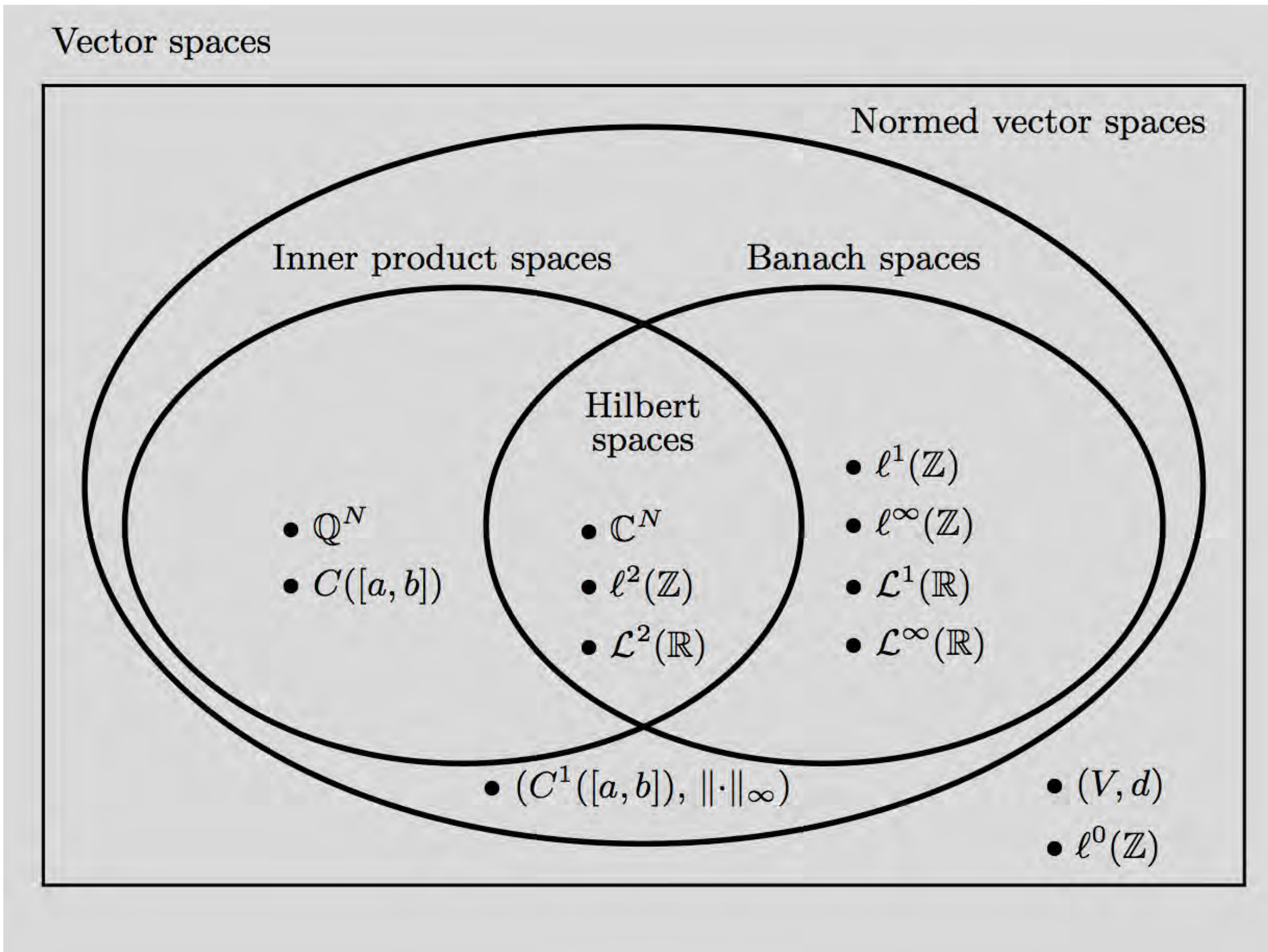
$$\|v - x_k\| < \varepsilon \qquad \text{for all} \quad k > K_\varepsilon.$$

DEFINITION 2.14 (CLOSED SUBSPACE) A subspace S of a normed vector space V is called *closed* when it contains all limits of sequences of vectors in S.

DEFINITION 2.15 (CAUCHY SEQUENCE OF VECTORS) A sequence of vectors x_0, $x_1, \ldots$ in a normed vector space is called a *Cauchy sequence* when: Given any $\varepsilon > 0$, there exists a K_ε such that

$$\|x_k - x_m\| < \varepsilon \qquad \text{for all } k,\, m > K_\varepsilon.$$

DEFINITION 2.16 (COMPLETENESS AND HILBERT SPACE) A normed vector space V is said to be *complete* when every Cauchy sequence in V converges to a vector in V. A complete inner product space is called a *Hilbert space*.



Exercises

2.20. *Closed subspaces and* $\ell^0(\mathbb{Z})$
Let $\ell^0(\mathbb{Z})$ denote the set of complex-valued sequences with a finite number of nonzero entries.

(i) Show that $\ell^0(\mathbb{Z})$ is a subspace of $\ell^2(\mathbb{Z})$.
(ii) Show that $\ell^0(\mathbb{Z})$ is not a closed subspace of $\ell^2(\mathbb{Z})$.

Exercise 2.20

(i) $\forall\, v \in \ell^0(\mathbb{Z})$, be $I = \{ i \mid v_i \neq 0 \}$ with $|I| = M < \infty$

M = number of non zero elements in v

Then $\|x\|_2 = \left(\sum_{i \in \mathbb{Z}} |v_i|^2 \right)^{1/2} = \left(\sum_{i \in I} |v_i|^2 \right)^{1/2} \leq$

$\leq \left(M \max_{i \in I} |v_i|^2 \right)^{1/2} = \sqrt{M} \max_{i \in I} |v_i| < \infty$

$\Rightarrow v \in \ell^2(\mathbb{Z}) \Rightarrow \ell^0(\mathbb{Z}) \subset \ell^2(\mathbb{Z})$

(ii) consider the sequence $v^{(M)} = [\cdots 0 \ 1 \ \frac{1}{2} \cdots \frac{1}{M} \ 0 \cdots] \in \ell^0(\mathbb{Z})$

let also $v = \lim_{M \to \infty} v^{(M)} \longrightarrow v \notin \ell^0(\mathbb{Z})$

However $v \in \ell^2(\mathbb{Z})$ since $\|v\|_2 = \left(\sum_{i=1}^{\infty} \frac{1}{i^2} \right)^{1/2} = \frac{\pi}{\sqrt{6}} < \infty$

$\Rightarrow \ell^0(\mathbb{Z})$ is not a closed subspace of $\ell^2(\mathbb{Z})$

Exercises

2.22. *Completeness*

Let $\mathcal{P}$ be the inner product space of polynomials with

$$\langle p, q \rangle \;=\; \int_0^1 p(t)\, q^*(t)\, dt,$$

and let (p_k) be a Cauchy sequence in $\mathcal{P}$,

$$p_k(t) \;=\; \sum_{i=0}^{k} \frac{1}{2^i}\, t^i .$$

Prove that $\mathcal{P} \subset \mathcal{L}^2([0,1])$ is not a Hilbert space.

Exercise 2.22

Indicating $p(t) = \lim_{k \to \infty} p_k(t) = \lim_{k \to \infty} \sum_{i=0}^{k} \frac{1}{2^i} t^i$

we have that $p(t) = \dfrac{1}{1 - \frac{1}{2}t}$ $\quad 0 \le t \le 1$

$\Rightarrow$ while $(p_k(t))$ is a Cauchy sequence in $\mathcal{P}$, it does not converge to an element in $\mathcal{P}$ since $p(t)$ is not a polynomial

$\Rightarrow$ $\mathcal{P}$ is an "inner product space", but since is not complete, it is not a Hilbert space

Exercises

2.24. *Cauchy sequences*

Show that in a normed vector space, every convergent sequence is a Cauchy sequence.

Let (x_n) a sequence convergent to x, $(x_n) \to x$

i.e. $\lim\limits_{n\to\infty} \| x - x_n \| = 0$

Now consider a Cauchy sequence, then

$$\lim_{n,m\to\infty} \| x_n - x_m \| = \lim_{n,m\to\infty} \| x_n - x + x - x_m \| =$$

$$\leq \lim_{n\to\infty} \| x - x_n \| + \lim_{m\to\infty} \| x - x_m \|$$

$$= 0$$

$\Rightarrow$ Every convergent sequence is a Cauchy sequence.