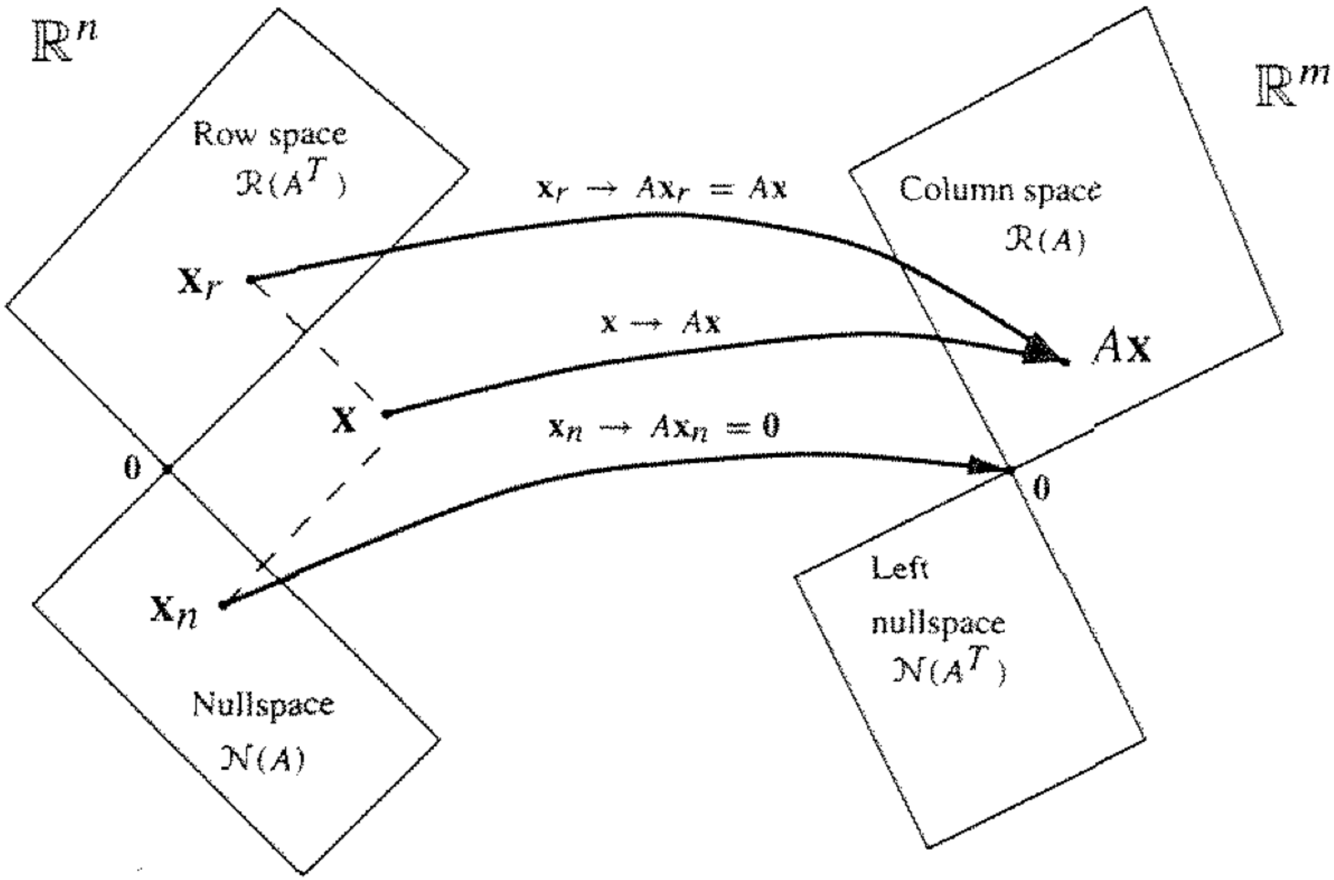
AMIR Exercises 2025/26

Linear Operators

Projections

Bases

Matrix operator algebra



The four fundamental subspaces of a matrix operator

Linear operators

DEFINITION 2.17 (LINEAR OPERATOR) A function $A : H_0 \to H_1$ is called a *linear operator* from H_0 to H_1 when, for all x, y in H_0 and α in $\mathbb{C}$ (or $\mathbb{R}$):

(i) *Additivity:* $A(x + y) = Ax + Ay$.

(ii) *Scalability:* $A(\alpha x) = \alpha(Ax)$.

When domain H_0 and codomain H_1 are the same, A is also called a linear operator on H_0.

DEFINITION 2.18 (OPERATOR NORM AND BOUNDED LINEAR OPERATOR) The *operator norm* of A, denoted by $\|A\|$, is defined as

$$\|A\| = \sup_{\|x\|=1} \|Ax\|. \tag{2.45}$$

A linear operator is called *bounded* when its operator norm is finite.

Linear operators

DEFINITION 2.19 (INVERSE) A bounded linear operator $A : H_0 \to H_1$ is called *invertible* if there exists a bounded linear operator $B : H_1 \to H_0$ such that

$$
\begin{aligned}
B\,Ax &= x, \quad \text{for every } x \text{ in } H_0, \quad \text{and} &\text{(2.46a)}\\
A\,By &= y, \quad \text{for every } y \text{ in } H_1. &\text{(2.46b)}
\end{aligned}
$$

When such a B exists, it is unique, is denoted by A^{-1}, and is called the *inverse* of A; B is called a *left inverse* of A when (2.46a) holds, and B is called a *right inverse* of A when (2.46b) holds.

DEFINITION 2.20 (ADJOINT AND SELF-ADJOINT OPERATOR) The linear operator $A^* : H_1 \to H_0$ is called the *adjoint* of the linear operator $A : H_0 \to H_1$ when

$$\langle Ax, y\rangle_{H_1} = \langle x, A^*y\rangle_{H_0}, \qquad \text{for every } x \text{ in } H_0 \text{ and } y \text{ in } H_1. \qquad (2.48)$$

When $A = A^*$, the operator A is called *self-adjoint* or *Hermitian*.

Linear operators

THEOREM 2.21 (ADJOINT PROPERTIES) Let $A : H_0 \to H_1$ be a bounded linear operator.

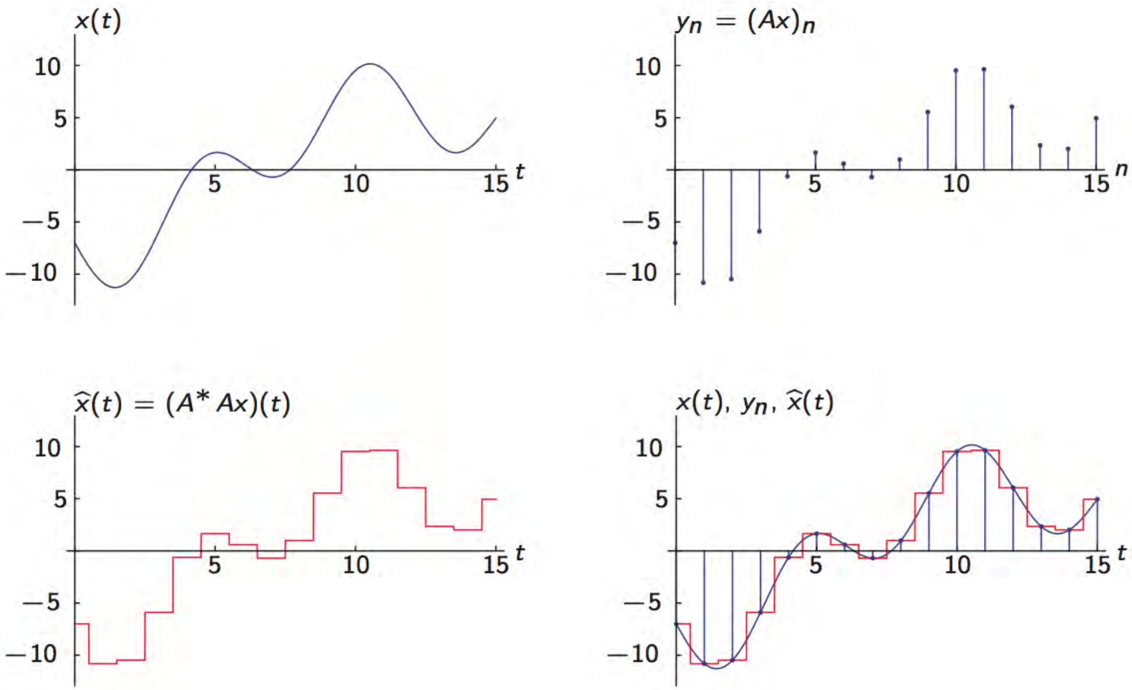
(i) The adjoint A^*, defined through (2.48), exists.
(ii) The adjoint A^* is unique.
(iii) The adjoint of A^* equals the original operator, $(A^*)^* = A$.
(iv) The operators AA^* and A^*A are self-adjoint.
(v) The operator norms of A and A^* are equal, $\|A^*\| = \|A\|$.
(vi) If A is invertible, the adjoint of the inverse and the inverse of the adjoint are equal, $(A^{-1})^* = (A^*)^{-1}$.
(vii) Let $B : H_0 \to H_1$ be a bounded linear operator. Then $(A + B)^* = A^* + B^*$.
(viii) Let $B : H_1 \to H_2$ be a bounded linear operator. Then $(BA)^* = A^*B^*$.

Adjoint operators: Local averaging

LAB

$$A : \mathcal{L}^2(\mathbb{R}) \to \ell^2(\mathbb{Z}) \qquad (Ax)_k = \int_{k-\frac{1}{2}}^{k+\frac{1}{2}} x(t)dt$$

$$\begin{aligned}\langle Ax, y\rangle_{\ell^2} &= \sum_{n\in\mathbb{Z}} (Ax)_n\, y_n^* = \sum_{n\in\mathbb{Z}} \left(\int_{n-1/2}^{n+1/2} x(t)\, dt\right) y_n^* = \sum_{n\in\mathbb{Z}} \int_{n-1/2}^{n+1/2} x(t) y_n^*\, dt \\ &= \sum_{n\in\mathbb{Z}} \int_{n-1/2}^{n+1/2} x(t)\widehat{x}^*(t)\, dt = \int_{-\infty}^{\infty} x(t)\widehat{x}^*(t)\, dt = \langle x, \widehat{x}\rangle_{\mathcal{L}^2} = \langle x, A^* y\rangle_{\mathcal{L}^2}\end{aligned}$$



!! The adjoint A* can be interpreted as the operator that takes a sequence and produce a staircase function

!! The adjoint A* **is not** the inverse

Linear operators

DEFINITION 2.22 (UNITARY OPERATOR) A bounded linear operator $A : H_0 \to H_1$ is called *unitary* when

(i) it is *invertible*; and

(ii) it *preserves inner products*,

$$\langle Ax, Ay \rangle_{H_1} = \langle x, y \rangle_{H_0} \qquad \text{for every } x, y \text{ in } H_0. \tag{2.54}$$

THEOREM 2.23 (UNITARY LINEAR OPERATOR) A bounded linear operator $A : H_0 \to H_1$ is unitary if and only if $A^{-1} = A^*$.

Linear operators

Proof. Condition (2.54) is equivalent to A^* being a left inverse of A:

$$A^* A = I \quad \text{on} \quad H_0. \tag{2.56a}$$

To see that (2.54) implies (2.56a), note that

$$\langle A^* A x,\, y\rangle \overset{(a)}{=} \langle Ax,\, Ay\rangle \overset{(b)}{=} \langle x,\, y\rangle,$$

where (a) follows from the definition of adjoint; and (b) from (2.54). Conversely, to see that (2.56a) implies (2.54), note that

$$\langle Ax,\, Ay\rangle \overset{(a)}{=} \langle x,\, A^* A y\rangle \overset{(b)}{=} \langle x,\, y\rangle,$$

where (a) follows from the definition of adjoint; and (b) from (2.56a).

Combining (2.54) with invertibility gives that A^* is a right inverse of A:

$$A A^* = I \quad \text{on} \quad H_1. \tag{2.56b}$$

To verify (2.56b), note that for every x, y in H_1,

$$\langle AA^* x,\, y\rangle = \langle AA^* x,\, AA^{-1} y\rangle \overset{(a)}{=} \langle A^* x,\, A^{-1} y\rangle \overset{(b)}{=} \langle x,\, AA^{-1} y\rangle = \langle x,\, y\rangle,$$

where (a) follows from (2.54); and (b) from the definition of adjoint.

The desired equivalence follows: If A is unitary, then A is invertible and (2.54) holds, so both conditions (2.56) hold, so $A^{-1} = A^*$. Conversely, if $A^{-1} = A^*$, then A is invertible and (2.56a) holds, implying (2.54).

Linear operators

DEFINITION 2.24 (EIGENVECTOR OF A LINEAR OPERATOR) An *eigenvector* of a linear operator $A : H \to H$ is a nonzero vector $v \in H$ such that

$$Av = \lambda v, \tag{2.57}$$

for some $\lambda \in \mathbb{C}$. The constant λ is called the corresponding *eigenvalue* and (λ, v) is called an *eigenpair*.

DEFINITION 2.25 (DEFINITE LINEAR OPERATOR) A self-adjoint operator $A : H \to H$ is called

(i) *positive semidefinite* or *nonnegative definite*, written $A \geq 0$, when

$$\langle Ax, x \rangle \geq 0 \quad \text{for all} \quad x \in H; \tag{2.58a}$$

(ii) *positive definite*, written $A > 0$, when

$$\langle Ax, x \rangle > 0 \quad \text{for all nonzero} \quad x \in H; \tag{2.58b}$$

(iii) *negative semidefinite* or *nonpositive definite* when $-A$ is positive semidefinite; and

(iv) *negative definite* when $-A$ is positive definite.

Projections

THEOREM 2.26 (PROJECTION THEOREM) Let S be a closed subspace of Hilbert space H and let x be a vector in H.

(i) *Existence:* There exists $\widehat{x} \in S$ such that $\|x - \widehat{x}\| \leq \|x - s\|$ for all $s \in S$.

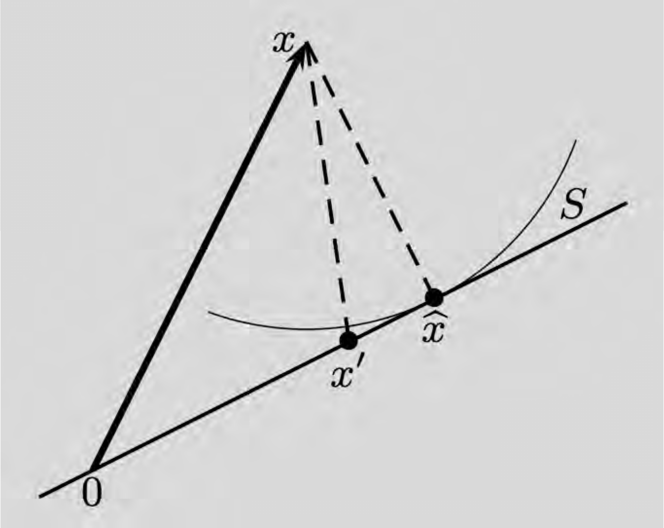
(ii) *Orthogonality:* $x - \widehat{x} \perp S$ is necessary and sufficient for determining $\widehat{x}$.

(iii) *Uniqueness:* The vector $\widehat{x}$ is unique.

(iv) *Linearity:* $\widehat{x} = Px$ where P is a linear operator that depends on S and not on x.

(v) *Idempotency:* $P(Px) = Px$ for all $x \in H$.

(vi) *Self-adjointness:* $P = P^*$.



Projections

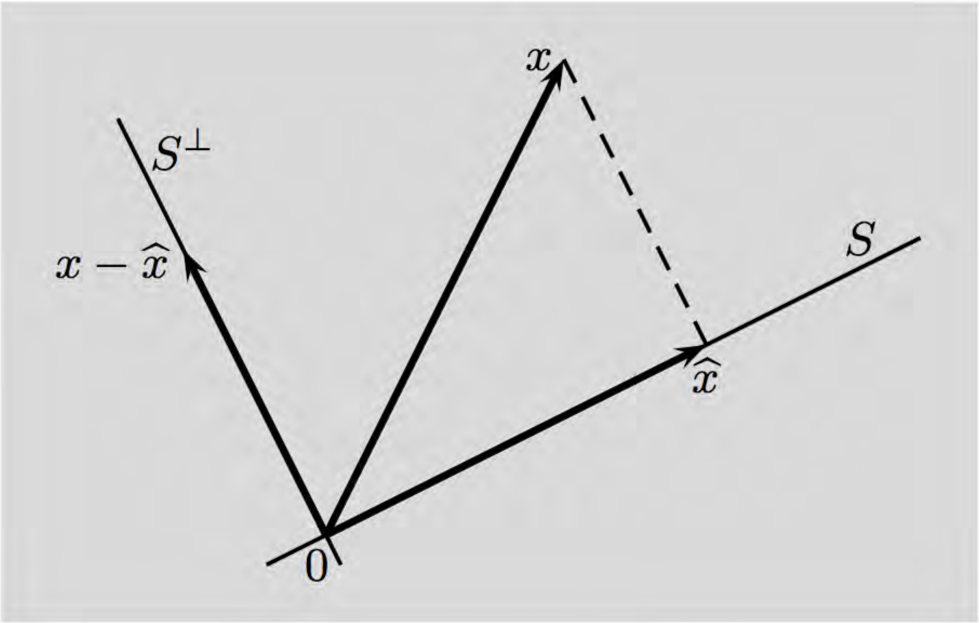


Figure 2.12 The best approximation of $x \in H$ within closed subspace S is uniquely determined by $x - \widehat{x} \perp S$. The solution generates an orthogonal decomposition of x into $\widehat{x} \in S$ and $x - \widehat{x} \in S^{\perp}$.

Projections

DEFINITION 2.27 (PROJECTION OPERATOR)

(i) An *idempotent* operator P is an operator such that $P^2 = P$.
(ii) A *projection* operator is a bounded linear operator that is idempotent.
(iii) An *orthogonal projection* operator is a projection operator that is self-adjoint.
(iv) An *oblique projection* operator is a projection operator that is not self-adjoint.

THEOREM 2.28 (ORTHOGONAL PROJECTION OPERATOR) A bounded linear operator P on a Hilbert space H satisfies

$$\langle x - Px, Py \rangle = 0 \qquad \text{for all } x, y \in H \tag{2.65}$$

if and only if P is an orthogonal projection operator.

Projections

Proof. Condition (2.65) is equivalent to having

$$0 = \langle x - Px, Py \rangle \overset{(a)}{=} \langle P^*(x - Px), y \rangle = \langle P^*(I - P)x, y \rangle \qquad \text{for all } x, y \in H,$$

where (a) follows from the definition of adjoint. This then implies that $P^*(I - P) = 0$, so

$$P^* = P^* P. \tag{2.66}$$

To show that this is equivalent to P being both idempotent and self-adjoint assume first that $P^* = P^* P$. Then,

$$P = (P^*)^* \overset{(a)}{=} (P^* P)^* = P^* P \overset{(b)}{=} P^*,$$

where (a) and (b) follow from (2.66); so P is self-adjoint. From this,

$$P^2 \overset{(a)}{=} P^* P \overset{(b)}{=} P^* \overset{(c)}{=} P,$$

where (a) and (c) follow from P being self-adjoint; and (b) from (2.66); so P is idempotent. Conversely, if P is idempotent and self-adjoint, then

$$P^* P \overset{(a)}{=} P^2 \overset{(b)}{=} P \overset{(c)}{=} P^*,$$

where (a) and (c) follow from P being self-adjoint; and (b) from P being idempotent.

Projections

THEOREM 2.29 (PROJECTION OPERATORS, ADJOINTS AND INVERSES) Let $A : H_0 \to H_1$ and $B : H_1 \to H_0$ be bounded linear operators. If A is a left inverse of B, then BA is a projection operator. If additionally $B = A^*$, then $BA = A^*A$ is an orthogonal projection operator.

EXAMPLE 2.23 (PROJECTION ONTO A SUBSPACE) Let

$$A = \tfrac{1}{2}\begin{bmatrix} 1 & 1 & -1 \\ -1 & 1 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Since A is a left inverse of B, we know from Theorem 2.29 that $P = BA$ is a projection operator. Explicitly,

$$P = BA = \frac{1}{2}\begin{bmatrix} 1 & 1 & -1 \\ 0 & 2 & 0 \\ -1 & 1 & 1 \end{bmatrix},$$

from which one can verify $P^2 = P$. A description of the 2-dimensional range of this projection operator is most transparent from B: it is the set of 3-tuples with middle component equal to the sum of the first and last (see Figure 2.14). Note that $P \neq P^*$, so the projection is oblique. Exercise 2.32 finds all A such that BA is a projection operator onto the range of B.

Projections

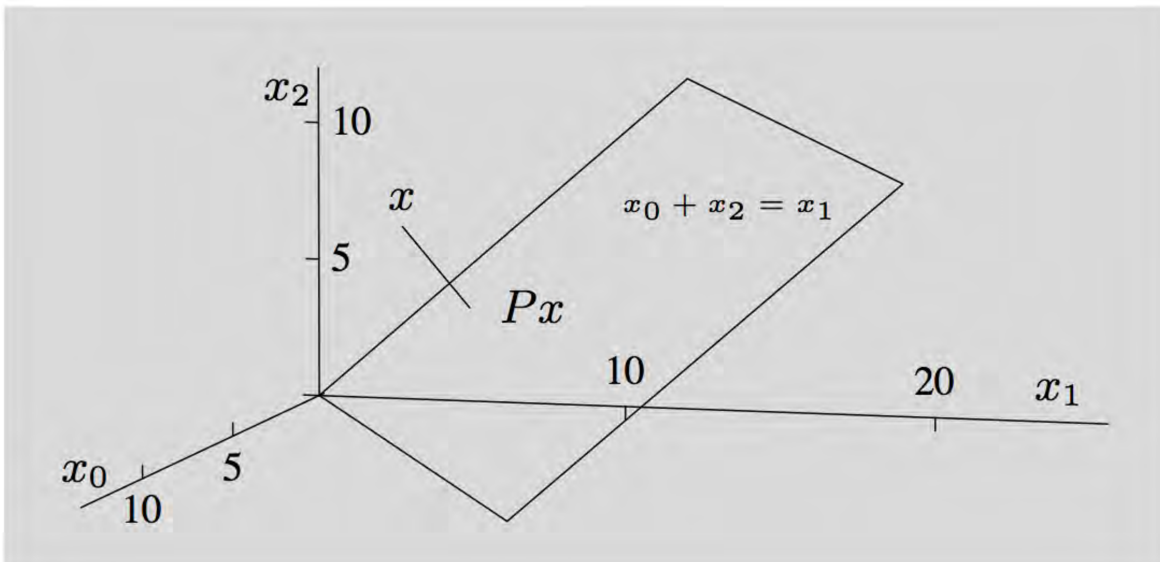


Figure 2.14 Two-dimensional range of the oblique projection operator P from Example 2.23. It is the plane $x_0 + x_2 = x_1$. For example, vector $x = [6\ 6\ 8]^\top$ is projected via P onto $Px = [2\ 6\ 4]$, not an orthogonal projection.

Projections

$$\frac{1}{2}\begin{bmatrix} 1 & 1 & -1 \\ 0 & 2 & 0 \\ -1 & 1 & 1 \end{bmatrix}\begin{bmatrix} x_0 \\ x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}x_0 + \frac{1}{2}x_1 - \frac{1}{2}x_3 \\ x_1 \\ -\frac{1}{2}x_0 + \frac{1}{2}x_1 + \frac{1}{2}x_3 \end{bmatrix}$$

$$\begin{bmatrix} x_0 \\ x_1 \\ x_2 \end{bmatrix} = x \qquad \begin{bmatrix} \cdots \end{bmatrix} = \hat{x}$$

$P : x \longmapsto \hat{x}$ to find locus of points where the generic x is projected

$$\begin{cases} (1)\quad x_0 = \hat{x}_0 = \frac{1}{2}x_0 + \frac{1}{2}x_1 - \frac{1}{2}x_2 \\ (2)\quad x_1 = \hat{x}_1 = x_1 \\ (3)\quad x_2 = \hat{x}_2 = -\frac{1}{2}x_0 + \frac{1}{2}x_1 + \frac{1}{2}x_2 \end{cases}$$

$$\xrightarrow{(1)+(2)} \boxed{x_0 + x_2 = x_1}$$

(proj. plane)

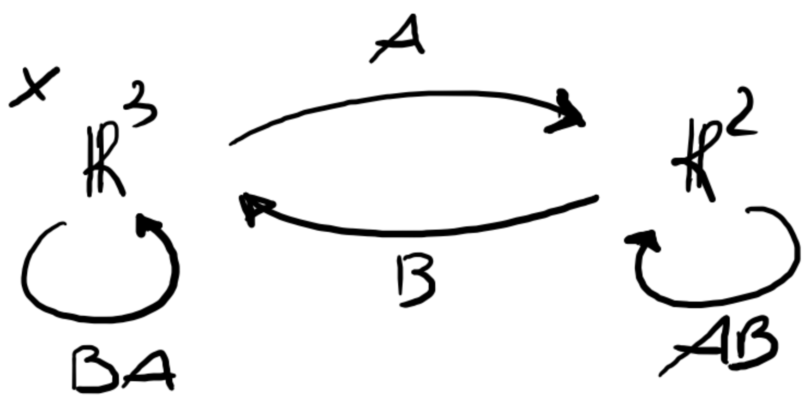
Exercises

2.32. *Projection operators*
Let

$$B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Find all projection operators onto the range of B; specify the orthogonal one.

In this case $H_0 = \mathbb{R}^3$ and $H_1 = \mathbb{R}^2$



if starting from $y \in \mathbb{R}^2$ $ABy = y$
then starting from $x \in \mathbb{R}^2$ $BAx = \hat{x}$
with $\hat{x} \in R(B) \subset \mathbb{R}^3$ and $BA\hat{x} = \hat{x}$

From theorem 2.29 we know that for $B : \mathbb{R}^2 \to \mathbb{R}^3$ and $A : \mathbb{R}^3 \to \mathbb{R}^2$ left inverse of B, $\Rightarrow BA$ is a projection operator. Moreover, if $(BA)^* = BA$ $\Rightarrow BA$ is an orthogonal projection operator.

First we find ALL possible left inverse of B. Thus we write

$$AB = \begin{bmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} = I$$

From this:

$a_{00} + a_{01} = 1$, $a_{01} + a_{02} = 0$, $a_{10} + a_{11} = 0$, $a_{11} + a_{12} = 1$

and calling $a_{01} = \alpha$ and $a_{11} = \beta$

$$A = \begin{bmatrix} 1-\alpha & \alpha & -\alpha \\ -\beta & \beta & 1-\beta \end{bmatrix}$$

Thus projection operators BA are

$$BA = \begin{bmatrix} 1-\alpha & \alpha & -\alpha \\ 1-(\alpha+\beta) & \alpha+\beta & 1-(\alpha+\beta) \\ -\beta & \beta & 1-\beta \end{bmatrix}$$

The orthogonal projection is the one for which

$$(BA)^* = BA \quad \longrightarrow \quad \begin{cases} \alpha = 1-(\alpha+\beta) \\ \alpha = \beta \end{cases} \quad \begin{cases} \alpha = 1-2\alpha \\ \alpha = \beta = \frac{1}{3} \end{cases}$$

$$\rightarrow A = \frac{1}{3}\begin{bmatrix} 2 & 1 & -1 \\ -1 & 1 & 2 \end{bmatrix} \text{ and } BA = \frac{1}{3}\begin{bmatrix} 2 & 1 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 2 \end{bmatrix}$$