AMIR Exercises 2025/26

Projections

Bases

Projections

THEOREM 2.30 (ORTHOGONAL PROJECTION VIA PSEUDOINVERSE) Let $A : H_0 \to H_1$ be a bounded linear operator.

(i) If AA^* is invertible, then

$$B = A^*(AA^*)^{-1} \tag{2.68a}$$

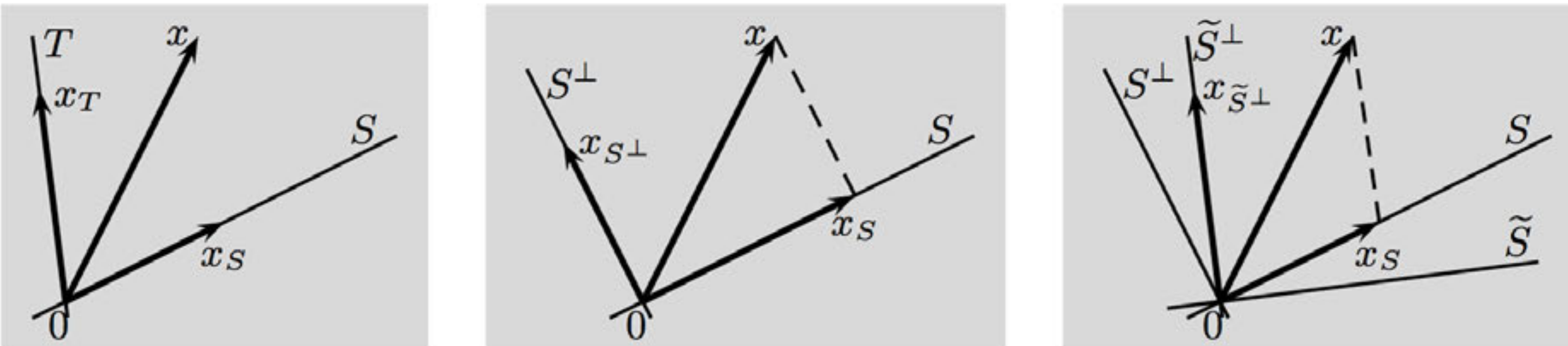
is the *pseudoinverse* of A, and $BA = A^*(AA^*)^{-1}A$ is the orthogonal projection operator onto the range of A^*.

(ii) If A^*A is invertible, then

$$B = (A^*A)^{-1}A^* \tag{2.68b}$$

is the *pseudoinverse* of A, and $AB = A(A^*A)^{-1}A^*$ is the orthogonal projection operator onto the range of A.

Projections



(a) Decomposition. (b) Orthogonal projection. (c) Oblique projection.

Figure 2.16 Decompositions and projections. (a) A vector space V is decomposed as a direct sum $S \oplus T$ when any $x \in V$ can be written uniquely as a sum of components in S and T. (b) An orthogonal projection operator generates an orthogonal direct sum decomposition of a Hilbert space. It decomposes vector x into $x_S \in S$ and $x_{S^\perp} \in S^\perp$. (c) An oblique projection operator generates a nonorthogonal direct sum decomposition of a Hilbert space. It decomposes vector x into $x_S \in S$ and $x_{\widetilde{S}^\perp} \in \widetilde{S}^\perp$.

Projections

DEFINITION 2.31 (DIRECT SUM AND DECOMPOSITION) A vector space V is a *direct sum* of subspaces S and T, denoted $V = S \oplus T$, when any nonzero vector $x \in V$ can be written uniquely as

$$x = x_S + x_T \qquad \text{where} \quad x_S \in S \quad \text{and} \quad x_T \in T. \tag{2.70}$$

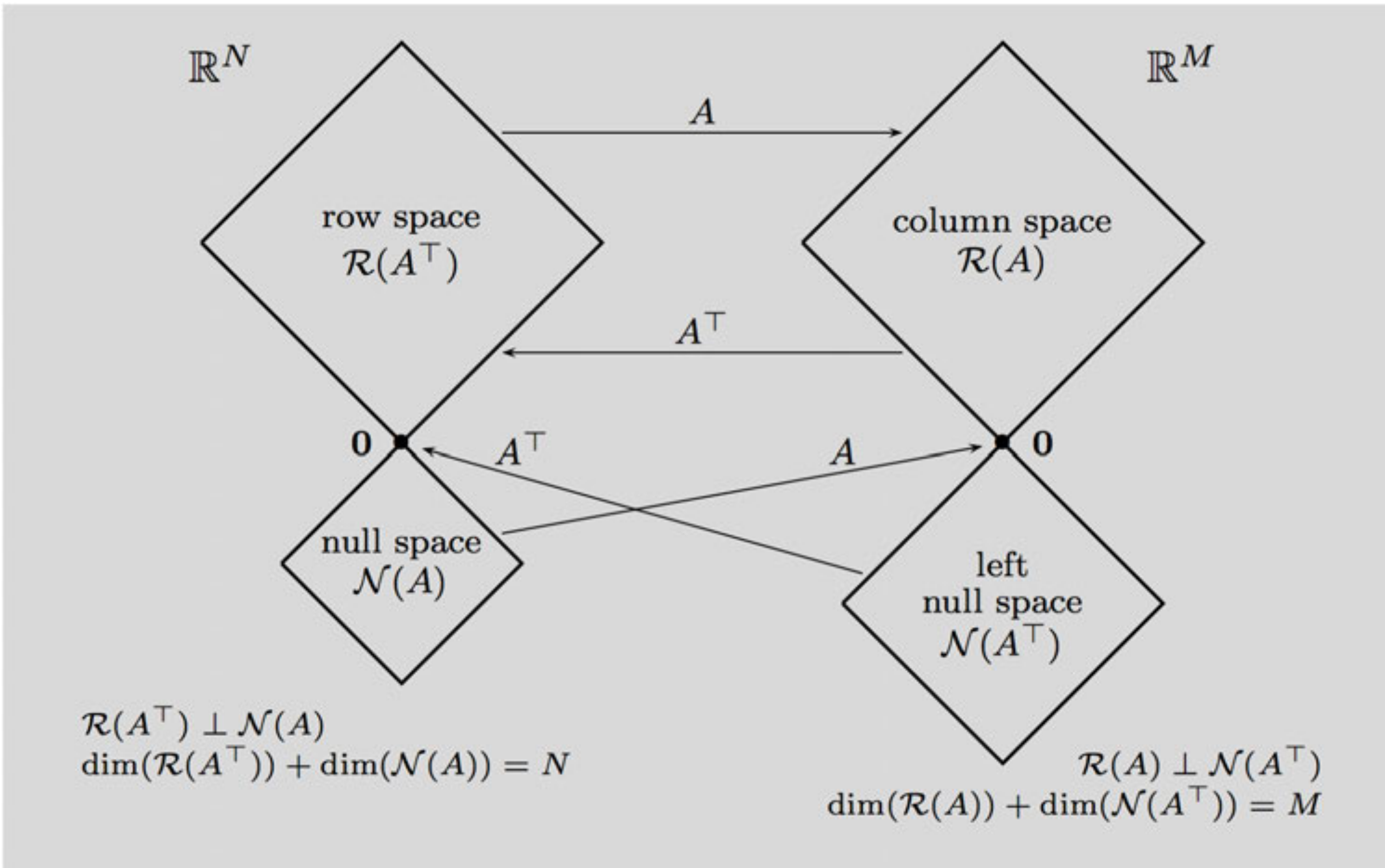
The subspaces S and T form a *decomposition* of V, and the vectors x_S and x_T form a decomposition of x. When S and T are orthogonal, this is called an *orthogonal decomposition*.

THEOREM 2.32 (DIRECT-SUM DECOMPOSITION FROM PROJECTION OPERATOR) Let H be a Hilbert space.

(i) Let P be a projection operator on H. It generates a direct-sum decomposition of H into its range $\mathcal{R}(P)$ and null space $\mathcal{N}(P)$: $H = S \oplus T$, where $S = \mathcal{R}(P)$ and $T = \mathcal{N}(P)$.

(ii) Conversely, let closed subspaces S and T satisfy $H = S \oplus T$. Then there exists a projection operator on H such that $S = \mathcal{R}(P)$ and $T = \mathcal{N}(P)$.

Matrix operator algebra



Projections

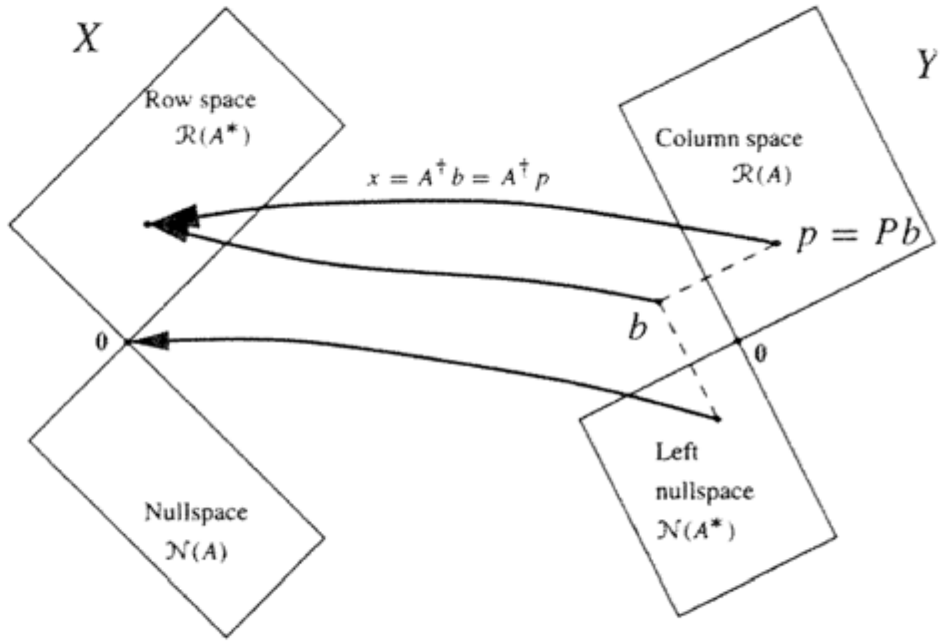


Figure 4.5: Operation of the pseudoinverse

$$x=(A^*A)^{-1}A^*p=(A^*A)^{-1}A^*Pb=(A^*A)^{-1}A^*A(A^*A)^{-1}A^*b=(A^*A)^{-1}A^*b$$

The operation of the pseudoinverse operator is shown in figure 4.5. The pseudoinverse operator takes a point from Y back to a point in $\hat{x} \in \mathcal{R}(A^*)$, in such a way that $\hat{x}$ has minimum norm. The operation of the pseudoinverse operation on a point $b \notin \mathcal{R}(A)$ is to first project b onto $\mathcal{R}(A)$ using the projection P, then to map back to $\mathcal{R}(A^*)$ to a vector $\hat{x}$ of minimum length; by this projection onto $\mathcal{R}(A)$ the error $b - A\hat{x}$ is minimized.

Bases

DEFINITION 2.34 (BASIS) The set of vectors $\Phi = \{\varphi_k\}_{k\in\mathcal{K}} \subset V$, where $\mathcal{K}$ is finite or countably infinite, is called a *basis* for a normed vector space V when

(i) it is *complete* in V, meaning for any $x \in V$, there is a sequence $\alpha \in \mathbb{C}^{\mathcal{K}}$ such that

$$x = \sum_{k\in\mathcal{K}} \alpha_k \varphi_k; \tag{2.86}$$

and

(ii) for any $x \in V$, the sequence α satisfying (2.86) is unique.

Bases

Definition 2.35 (Riesz basis) The set of vectors $\Phi = \{\varphi_k\}_{k \in \mathcal{K}} \subset H$, where $\mathcal{K}$ is finite or countably infinite, is called a *Riesz basis* for Hilbert space H when

(i) it is a *basis* for H; and

(ii) there exist *stability constants* $\lambda_{\min}$ and $\lambda_{\max}$ satisfying $0 < \lambda_{\min} \leq \lambda_{\max} < \infty$ such that, for any x in H, the expansion of x with respect to the basis Φ, $x = \sum_{k \in \mathcal{K}} \alpha_k \varphi_k$, satisfies

$$\lambda_{\min} \|x\|^2 \;\leq\; \sum_{k \in \mathcal{K}} |\alpha_k|^2 \;\leq\; \lambda_{\max} \|x\|^2. \tag{2.88}$$

The largest such $\lambda_{\min}$ and smallest such $\lambda_{\max}$ are called *optimal stability constants* of Φ.

Bases

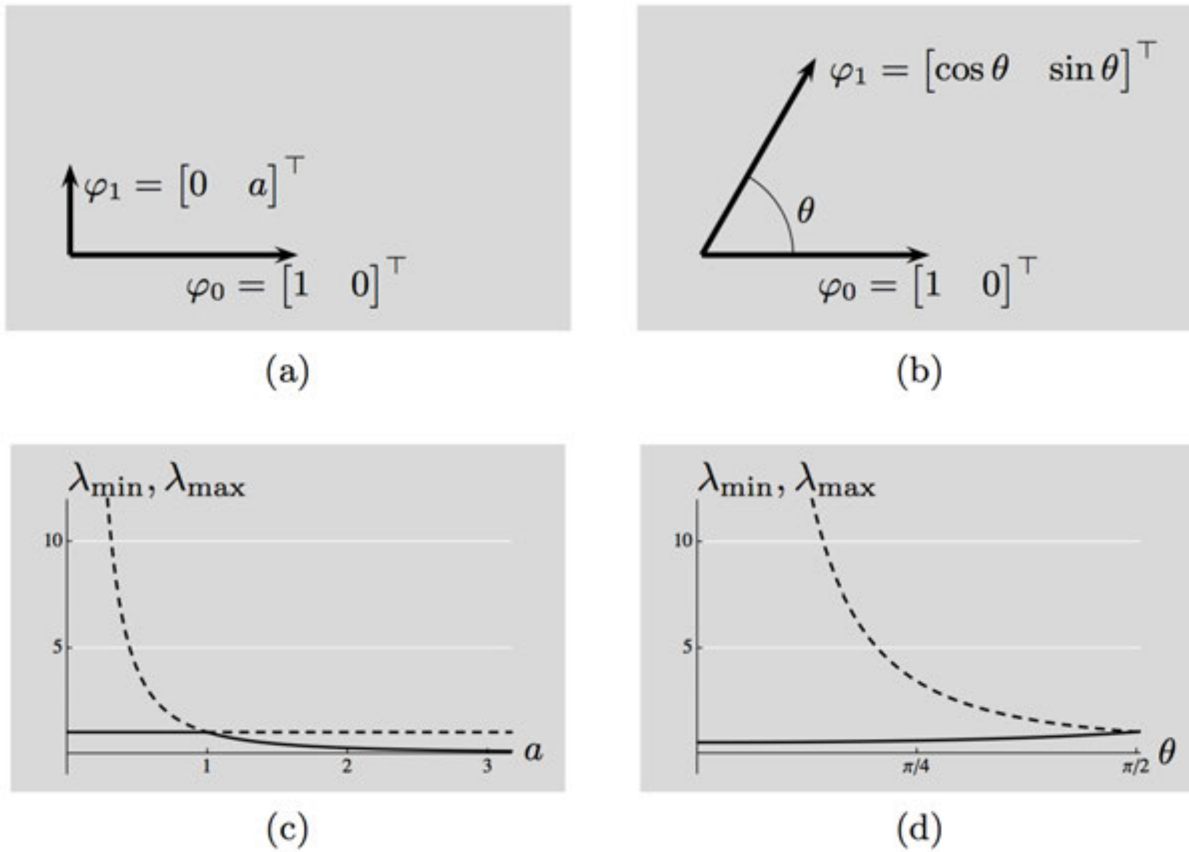


Figure 2.18 Two families of bases in $\mathbb{R}^2$ that deviate from the standard basis $\{e_0, e_1\}$ and their optimal stability constants $\lambda_{\min}$ (solid) and $\lambda_{\max}$ (dashed). (a) φ_1 is orthogonal to φ_0 but not necessarily of unit length. (b) φ_1 is of unit length but not necessarily orthogonal to φ_0. (c) $\lambda_{\min}$ and $\lambda_{\max}$ for the basis in (a) as a function of a. (d) $\lambda_{\min}$ and $\lambda_{\max}$ for the basis in (b) as a function of θ.

Exercises

2.33. *Riesz bases*

(i) Prove that the standard basis in $\ell^2(\mathbb{Z})$ is a Riesz basis with optimal stability constants $\lambda_{\min} = \lambda_{\max} = 1$.

(ii) Let $\{e_k\}_{k\in\mathbb{Z}}$ denote the standard basis in $\ell^2(\mathbb{Z})$ and define the following scaled version:

$$\varphi_k = 2^k e_k, \qquad k \in \mathbb{Z}.$$

Prove that $\{\varphi_k\}_{k\in\mathbb{Z}}$ is a basis, but there is neither a positive $\lambda_{\min}$ nor a finite $\lambda_{\max}$ such that (2.88) in the definition of Riesz basis holds.

(iii) Let

$$\psi_k = \cos(k)\, e_k, \qquad k \in \mathbb{Z}.$$

Prove or disprove that $\{\psi_k\}_{k\in\mathbb{Z}}$ is a basis for $\ell^2(\mathbb{Z})$, and prove or disprove that $\{\psi_k\}_{k\in\mathbb{Z}}$ is a Riesz basis for $\ell^2(\mathbb{Z})$.

(ii) the standard basis for $\ell^2(\mathbb{Z})$ is $\{e_k\}_{k\in\mathbb{Z}}$

with $e_k = [\cdots\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ \cdots]$

↑ k-th position

$\Rightarrow \alpha_k = \langle x, e_k \rangle$ → we need to bound

$$\sum_{k\in\mathbb{Z}} |\alpha_k|^2 = \sum_{k\in\mathbb{Z}} |\langle x, e_k\rangle|^2 = \sum_{k\in\mathbb{Z}} |x_k|^2 = \|x\|^2$$

$\Rightarrow$ the standard basis is a Riesz basis with

$\lambda_{min} = \lambda_{max} = 1$

(ii) the set φ_k of scaled e_k remains a basis since the vectors are still linearly independent

We can write

$$x = \sum_{k \in \mathbb{Z}} x_k e_k = \sum_{k \in \mathbb{Z}} \underbrace{x_k 2^{-k}}_{\alpha_k =} \varphi_k = \sum_{k \in \mathbb{Z}} \alpha_k \varphi_k$$

$\alpha_k =$ expansion coefficients for $\{\varphi_k\}$

$\Rightarrow$ we try to bound $\sum_k |\alpha_k|^2$

Let $x = e_m$ for some $m \in \mathbb{Z}$ $\Rightarrow$ there is only one expansion coefficient $\alpha_m = 2^{-m}$, and $x = \alpha_m \varphi_m = 2^{-m} \cdot 2^m e_m$

Thus $\sum_k |\alpha_k|^2 = 2^{-2m}$. Since m is an arbitrary integer

$m \to \infty$ shows that there is no positive bound λ_{min}

and $m \to -\infty$ shows that there is no finite λ_{max} satisfying the Riesz basis condition $0 < \lambda_{min} \leq 2^{-2m} \leq \lambda_{max} < \infty$, $\forall m$

(iii) $\psi_k = \cos(k)\, e_k$ $\quad$ ψ_k Basis? $\quad$ ψ_k Riesz basis?

Since $0 < |\cos(k)| \leq 1 \quad \forall k \in \mathbb{Z}$ => vectors are still linear independent and multiplied by a non zero coefficients => ψ_k is still a basis for $\ell^2(\mathbb{Z})$

with $x = \sum_{k\in\mathbb{Z}} x_k e_k = \sum_{k\in\mathbb{Z}} \underbrace{x_k \frac{1}{\cos k}}_{\text{expansion coefficients}} \psi_k = \sum_{k\in\mathbb{Z}} \beta_k \psi_k$

↳ we try to bound $\sum_k |\beta_k|^2$

Since $0 < \cos^2 k \leq 1$

$$\sum_{k\in\mathbb{Z}} |\beta_k|^2 = \sum_{k\in\mathbb{Z}} \left|\frac{1}{\cos k} x_k\right|^2 = \sum_{k\in\mathbb{Z}} \frac{1}{\cos^2 k} |x_k|^2 \geq \sum_{k\in\mathbb{Z}} |x_k|^2 = \|x\|^2$$

=> the lower bound for stability of the basis is $\lambda_{min} = 1$

But there is no upper bound $\lambda_{max} < \infty$ since $\cos^2 k$ can be arbitrarily close to zero.

Bases

DEFINITION 2.36 (BASIS SYNTHESIS OPERATOR) Given a Riesz basis $\{\varphi_k\}_{k\in\mathcal{K}}$ for Hilbert space H, the *synthesis operator* associated with it is

$$\Phi : \ell^2(\mathcal{K}) \rightarrow H, \qquad \text{with} \qquad \Phi\,\alpha \;=\; \sum_{k\in\mathcal{K}} \alpha_k \varphi_k. \tag{2.89}$$

DEFINITION 2.37 (BASIS ANALYSIS OPERATOR) Given a Riesz basis $\{\varphi_k\}_{k\in\mathcal{K}}$ for Hilbert space H, the *analysis operator* associated with it is

$$\Phi^* : H \rightarrow \ell^2(\mathcal{K}), \qquad \text{with} \qquad (\Phi^*\,x)_k \;=\; \langle x, \varphi_k\rangle, \quad k \in \mathcal{K}. \tag{2.90}$$

Bases

DEFINITION 2.38 (ORTHONORMAL BASIS) The set of vectors $\Phi = \{\varphi_k\}_{k\in\mathcal{K}} \subset H$, where $\mathcal{K}$ is finite or countably infinite, is called an *orthonormal basis* for the Hilbert space H when

(i) it is a *basis* for H; and

(ii) it is *orthonormal*,

$$\langle \varphi_i, \varphi_k \rangle = \delta_{i-k} \qquad \text{for every } i, k \in \mathcal{K}. \tag{2.91}$$

Bases

THEOREM 2.39 (ORTHONORMAL BASIS EXPANSIONS) Let $\Phi = \{\varphi_k\}_{k \in \mathcal{K}}$ be an orthonormal basis for Hilbert space H. The unique expansion with respect to Φ of any x in H has expansion coefficients

$$\alpha_k = \langle x, \varphi_k \rangle \qquad \text{for } k \in \mathcal{K}, \quad \text{or,} \tag{2.92a}$$

$$\alpha = \Phi^* x. \tag{2.92b}$$

Synthesis with these coefficients yields

$$x = \sum_{k \in \mathcal{K}} \langle x, \varphi_k \rangle \varphi_k \tag{2.93a}$$

$$= \Phi \alpha = \Phi \Phi^* x. \tag{2.93b}$$

Bases

THEOREM 2.40 (PARSEVAL'S EQUALITIES) Let $\Phi = \{\varphi_k\}_{k\in\mathcal{K}}$ be an orthonormal basis for Hilbert space H. Expansion with coefficients (2.92) satisfies *Parseval's equality*,

$$\|x\|^2 = \sum_{k\in\mathcal{K}} |\langle x, \varphi_k\rangle|^2 \tag{2.95a}$$

$$= \|\Phi^* x\|^2 = \|\alpha\|^2, \tag{2.95b}$$

and the *generalized Parseval's equality*,

$$\langle x, y\rangle = \sum_{k\in\mathcal{K}} \langle x, \varphi_k\rangle \langle y, \varphi_k\rangle^* \tag{2.96a}$$

$$= \langle \Phi^* x, \Phi^* y\rangle = \langle \alpha, \beta\rangle. \tag{2.96b}$$

Bases

THEOREM 2.41 (ORTHOGONAL PROJECTION ONTO A SUBSPACE) Given an orthonormal set $\Phi = \{\varphi_k\}_{k \in \mathcal{I}} \subset H$,

$$P_{\mathcal{I}}\, x = \sum_{k \in \mathcal{I}} \langle x, \varphi_k \rangle \varphi_k \tag{2.105a}$$

$$= \Phi_{\mathcal{I}}\, \Phi_{\mathcal{I}}^*\, x \tag{2.105b}$$

is the orthogonal projection of x onto $S_{\mathcal{I}} = \overline{\text{span}}(\{\varphi_k\}_{k \in \mathcal{I}})$.

THEOREM 2.42 (BESSEL'S INEQUALITY) Given an orthonormal set $\Phi = \{\varphi_k\}_{k \in \mathcal{I}}$ in a Hilbert space H, *Bessel's inequality* holds:

$$\|x\|^2 \geq \sum_{k \in \mathcal{I}} |\langle x, \varphi_k \rangle|^2 \tag{2.108a}$$

$$= \|\Phi_{\mathcal{I}}^*\, x\|^2. \tag{2.108b}$$

Equality for every x in H implies that the set Φ is complete in H, so the orthonormal set is an orthonormal basis for H; (2.108) is then Parseval's equality (2.95).

Bases

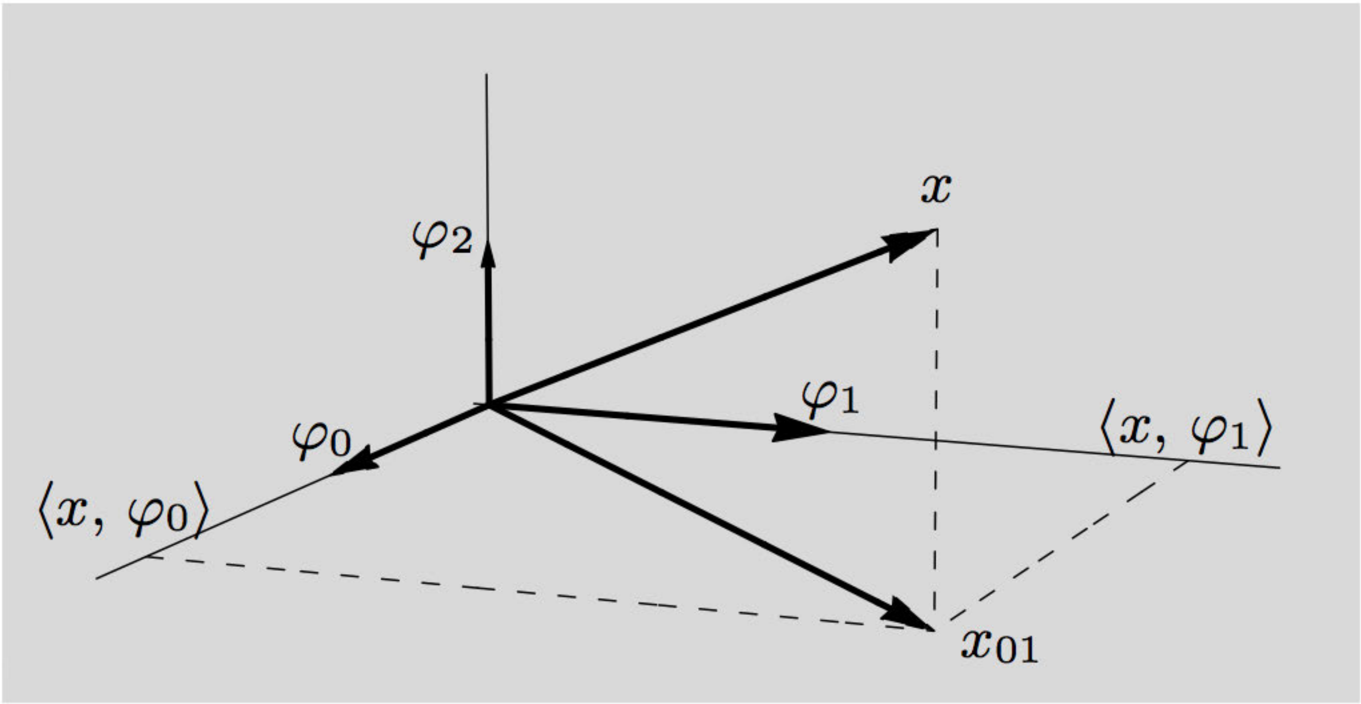


Figure 2.20 Illustration of Bessel's inequality in $\mathbb{R}^3$.

Biorthogonal (dual) Bases

DEFINITION 2.43 (BIORTHOGONAL PAIR OF BASES) The sets of vectors $\Phi = \{\varphi_k\}_{k\in\mathcal{K}} \subset H$ and $\widetilde{\Phi} = \{\widetilde{\varphi}_k\}_{k\in\mathcal{K}} \subset H$, where $\mathcal{K}$ is finite or countably infinite, are called a *biorthogonal pair of bases* for Hilbert space H when

(i) each is a *basis* for H; and

(ii) they are *biorthogonal*, meaning

$$\langle \varphi_i, \widetilde{\varphi}_k \rangle = \delta_{i-k} \qquad \text{for every } i, k \in \mathcal{K}. \tag{2.110}$$

THEOREM 2.44 (BIORTHOGONAL BASIS EXPANSIONS) Let $\Phi = \{\varphi_k\}_{k\in\mathcal{K}}$ and $\widetilde{\Phi} = \{\widetilde{\varphi}_k\}_{k\in\mathcal{K}}$ be a biorthogonal pair of bases for Hilbert space H. The unique expansion with respect to the basis Φ of any x in H has expansion coefficients

$$\alpha_k = \langle x, \widetilde{\varphi}_k \rangle \qquad \text{for } k \in \mathcal{K}, \quad \text{or,} \tag{2.112a}$$

$$\alpha = \widetilde{\Phi}^* x. \tag{2.112b}$$

Biorthogonal (dual) Bases

$a_1 = [2 \quad 1]^T$
$a_2 = [1 \quad 2]^T$

$x = [1 \quad 1]^T \qquad x = \alpha_1 a_1 + \alpha_2 a_2$

$\alpha_1 = \langle x, b_1 \rangle / \langle a_1, b_1 \rangle = 1/3$

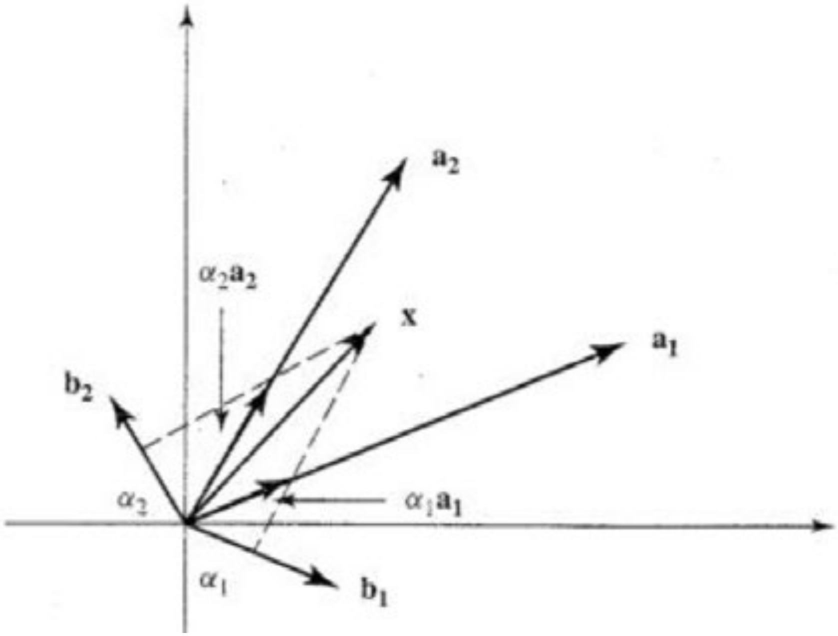
$\alpha_2 = \langle x, b_2 \rangle / \langle a_2, b_2 \rangle = 1/3$

a_1-a_2 and b_1-b_2 are biorthogonal

$\langle a_1, b_1 \rangle = 1$
$\langle a_2, b_2 \rangle = 1$
$\langle a_1, b_2 \rangle = 0$
$\langle a_2, b_1 \rangle = 0$

Dual Bases

$b_1 = [2/3 \quad -1/3]^T$
$b_2 = [-1/3 \quad 2/3]^T$



Biorthogonal (dual) Bases

Synthesis with these coefficients yields

$$x = \sum_{k \in \mathcal{K}} \langle x, \widetilde{\varphi}_k \rangle \varphi_k \tag{2.113a}$$

$$= \Phi \alpha = \Phi \widetilde{\Phi}^* x. \tag{2.113b}$$

THEOREM 2.45 (PARSEVAL'S EQUALITIES FOR BIORTHOGONAL PAIRS OF BASES) Let $\Phi = \{\varphi_k\}_{k \in \mathcal{K}}$ and $\widetilde{\Phi} = \{\widetilde{\varphi}_k\}_{k \in \mathcal{K}}$ be a biorthogonal pair of bases for Hilbert space H. Expansion with respect to the bases Φ and $\widetilde{\Phi}$ with coefficients (2.112) and (2.114) satisfies

$$\|x\|^2 = \sum_{k \in \mathcal{K}} \langle x, \varphi_k \rangle \langle x, \widetilde{\varphi}_k \rangle^* \tag{2.117a}$$

$$= \langle \Phi^* x, \widetilde{\Phi}^* x \rangle = \langle \widetilde{\alpha}, \alpha \rangle. \tag{2.117b}$$

Biorthogonal (dual) Bases

THEOREM 2.46 (DUAL BASIS) Let $\Phi = \{\varphi_k\}_{k\in\mathcal{K}}$ be a Riesz basis for Hilbert space H, and let $A : \ell^2(\mathcal{K}) \to \ell^2(\mathcal{K})$ be the inverse of the Gram matrix of Φ, that is, $A = (\Phi^*\Phi)^{-1}$. Then the set $\widetilde{\Phi} = \{\widetilde{\varphi}_k\}_{k\in\mathcal{K}}$ defined via

$$\widetilde{\varphi}_k = \sum_{\ell\in\mathcal{K}} a_{\ell,k}\varphi_\ell, \qquad \text{for each } k \in \mathcal{K}, \tag{2.130a}$$

together with Φ forms a biorthogonal pair of bases for H. The synthesis operator for this basis is given by

$$\widetilde{\Phi} = \Phi A = \Phi(\Phi^*\Phi)^{-1}, \tag{2.130b}$$

the pseudoinverse of Φ^*.

Exercises

2.39. *Dual bases*

Let $\Phi = \{\varphi_k\}_{k \in \mathcal{K}}$ be a Riesz basis for Hilbert space H with optimal stability constants $\lambda_{\min}$ and $\lambda_{\max}$.

(i) Show that the dual of the dual of Φ is Φ.

(ii) Show that the dual of Φ is Φ if and only if Φ is an orthonormal basis.

(iii) Show that the dual of Φ is a Riesz basis with optimal stability constants $1/\lambda_{\max}$ and $1/\lambda_{\min}$.

Exercises

(i) According to Theorem 2.46, given Φ, its unique dual is

$$\widetilde{\Phi} = \Phi(\Phi^*\Phi)^{-1}.$$

Its dual is then

$$\begin{aligned}\widetilde{\widetilde{\Phi}} &= \widetilde{\Phi}(\widetilde{\Phi}^*\widetilde{\Phi})^{-1} = \Phi(\Phi^*\Phi)^{-1}((\Phi(\Phi^*\Phi)^{-1})^*(\Phi(\Phi^*\Phi)^{-1}))^{-1} \\ &= \Phi(\Phi^*\Phi)^{-1}(((\Phi^*\Phi)^{-1})^*\Phi^*\Phi(\Phi^*\Phi)^{-1})^{-1} \\ &= \Phi(\Phi^*\Phi)^{-1}(((\Phi^*\Phi)^*)^{-1})^{-1} = \Phi(\Phi^*\Phi)^{-1}\Phi^*\Phi = \Phi.\end{aligned}$$

(ii) We can express the statement that the dual of Φ is Φ as

$$\Phi(\Phi^*\Phi)^{-1} = \Phi.$$

Since Φ is a basis by assumption, we can multiply both sides by the inverse of Φ,

$$\Phi^*\Phi = I,$$

or, in other words, Φ is unitary (orthonormal basis).

Exercises

(iii) We use (2.112b) to write (2.88) as follows:

$$\lambda_{\min}\, x^* x \;\le\; (\widetilde{\Phi}^*\, x)^* (\widetilde{\Phi}^*\, x) \;=\; x^* \widetilde{\Phi} \widetilde{\Phi}^* x \;\le\; \lambda_{\max}\, x^* x.$$

Similarly, for the dual basis, we want to bound $x^* \Phi \Phi^* x$. Since $\Phi\Phi^*$ is a positive definite matrix, according to (2.243), it can be bounded from below and above by its minimum and maximum eigenvalues. Because

$$\Phi\Phi^* \;=\; (\widetilde{\Phi}^*)^{-1} \widetilde{\Phi}^{-1} \;=\; (\widetilde{\Phi}\widetilde{\Phi}^*)^{-1},$$

and the eigenvalues of $\Phi\Phi^*$ and its inverse are inverses of each other,

$$\frac{1}{\lambda_{\max}}\, x^* x \;\le\; x^* \Phi \Phi^* x \;\le\; \frac{1}{\lambda_{\min}}\, x^* x.$$