Advanced methods for Information Representation

Vector spaces-I
Basics/Norm/Inner product/Distances

Riccardo Leonardi

Università degli Studi di Brescia

UNIVERSITÀ DEGLI STUDI DI BRESCIA

Department of Information Engineering
University of Brescia
Via Branze, 38 – 25231 Brescia - ITALY

Group: definition

- Group (G,*)

It is an algebraic structure where the operation "*" between the elements of G is

1. associative, i.e. $\forall$ a,b,c $\in$ G, (a * b) * c = a * (b * c)
2. $\exists$ a neutral element with respect to "*", i.e.

   $\forall a \in G, \exists e_* \in G, a * e_* = e_* * a = a$
3. Every element of G has an inverse with respect to "*", i.e.

   $\forall a \in G, \exists b \in G, a * b = b * a = e_*$

- If $\forall$ a,b $\in$ G, a * b = b * a, (G,*) is said to be commutative (or abelian)

Field: definition

- (K,+,.): algebraic structure over which 2 operations are defined, "+" and "." such that:
  - (K,+) forms an abelian group;
  - $(K\backslash\{e_{+}\},.)$ forms a group with neutral element $e_{.}$
  - "." is distributive with respect to +, i.e. ∀ a,b,c ∈ K, a.(b+c) = a.b+a.c

- ($\mathbb{C}$,+,.) represents the field of complex numbers, with "+" and "." being the addition/multiplication on complex numbers
- ($\mathbb{R}$,+,.) represents the field of real numbers, with "+" and "." being the addition/multiplication on real numbers

Vector spaces: definition

- V forms a vector space on the field of complex numbers $\mathbb{C}$ if
  1. (V, +) forms a commutative group, where + identifies the "sum" operation between the elements of V. Its neutral element is $\underline{0} \triangleq \underline{e}_+$.
  2. ∃ an external product "." between the elements of V and $\mathbb{C}$, for which
     a) complex number multiplication "." is interchangeable with respect to ".", i.e.

        $\forall$ a,b, $\in \mathbb{C}$, $\underline{v} \in V$, $(a.b).\underline{v} = a.(b.\underline{v})$

     b) $1 \in \mathbb{C}$ is a neutral element for ".", i.e. $\forall\ \underline{v} \in V$, $1.\underline{v} = \underline{v}$

     c) "." is distributive with respect to the sum "+" of the elements in V, i.e.

        $\forall$ a $\in \mathbb{C}$, $\underline{x},\underline{y} \in V$, $a.(\underline{x}+\underline{y}) = (a.\underline{x}) + (a.\underline{y})$

     d) "." is (sort of) distributive" with respect to the sum "+" defined over $\mathbb{C}$, i.e.

        $\forall$ a, b $\in \mathbb{C}$, $\underline{x} \in V$, $(a+b).\underline{x} = (a.\underline{x}) + (b.\underline{x})$

- The elements of V are called "vectors".

Vector spaces: examples

- Space $\mathbb{C}^N$ / $\mathbb{R}^N$ of complex-/(real-)valued finite dimensional vectors

  $\mathbb{C}^N / \mathbb{R}^N = \{ \underline{x} = [x_0 \; x_1 \; \ldots \; x_{N-1}]^T \mid x_n \in \mathbb{C}/\mathbb{R} \, , \, n = 0, 1, \ldots, N-1 \}$

  $\underline{x} + \underline{y} \quad \triangleq \quad [x_0+y_0 \;\; x_1+y_1 \; \ldots \; x_{N-1}+y_{N-1}]^T$

  $a.\underline{x} \quad \triangleq \quad [a.x_0 \;\; a.x_1 \; \ldots \; a.x_{N-1}]^T$

- Space $\mathbb{C}^{\mathbb{Z}}$ / $\mathbb{R}^{\mathbb{Z}}$ of complex-/(real-)valued infinite sequences

  $\mathbb{C}^{\mathbb{Z}} / \mathbb{R}^{\mathbb{Z}} = \{ \underline{x} = [\ldots \; x_{-1} \; x_0 \; x_1 \; \ldots]^T \mid x_n \in \mathbb{C}/\mathbb{R} \, , \, n \in \mathbb{Z} \}$

  $\underline{x} + \underline{y} \quad \triangleq \quad [\ldots \; x_{-1}+y_{-1} \;\; x_0+y_0 \;\; x_1+y_1 \; \ldots]^T$

  $a.\underline{x} \quad \triangleq \quad [\ldots \; a.x_{-1} \; a.x_0 \;\; a.x_1 \; \ldots \; a.x_{N-1}]^T$

Vector spaces: examples

- Space $\mathbb{C}^{\mathbb{R}}$ / $\mathbb{R}^{\mathbb{R}}$ of complex-/(real-)valued functions over $\mathbb{R}$

  $\mathbb{C}^{\mathbb{R}} / \mathbb{R}^{\mathbb{R}} = \{ \underline{x} = x(t) \mid x(t) \in \mathbb{C}/\mathbb{R}, t \in \mathbb{R} \}$

  $\underline{x} + \underline{y} \quad \hat{=} \quad (x+y)(t)$ sum of functions

  $a.\underline{x} \quad \hat{=} \quad (a.x)(t)$ external multiplication between a scalar and a function

- Space $\mathbb{C}^{\mathbb{R}^+}$ of complex-valued functions defined over $\mathbb{R}^+$

- Space $\mathbb{C}^{[a,b]}$ of complex-valued functions defined over [a,b]

- Space of polynomial functions of order N-1: $\underline{x} = \sum_{n=0}^{N-1} \alpha_n . t^n$

Subspace: definition/examples

- A non-empty subset S of a vector space V is called a subspace of V, when it is closed with respect to the operations of vector addition "+" and scalar multiplication ". ":
  1. $\forall\ \underline{x},\underline{y} \in S,\ \underline{x} + \underline{y} \in S$
  2. $\forall\ a \in \mathbb{C},\ \underline{x} \in S,\ a.\underline{x} \in S$

  (alternatively, $\forall\ a,b \in \mathbb{C},\ \underline{x},\underline{y} \in S,\ a.\underline{x} + b.\underline{y} \in S$)

- Examples of subspaces
  - $S_1 = \{\ \underline{x} = a.\underline{x}_0 \mid \text{fixed } \underline{x}_0 \in V,\ \forall a \in \mathbb{C}\ \}$
  - $S_2 = \{\ \underline{x} \in \mathbb{C}^{\mathbb{Z}} \mid x_n = 0,\ \forall n \neq 1,2,3\ \}$, subspace of sequences having 0 value for indices $n \neq 1,2,3$
  - $S_3 = \{\ \underline{x} \in \mathbb{C}^{\mathbb{R}} \mid x(t) = -x(-t)\ \}$, subspace of odd complex-valued functions

Affine subspace: definition/examples

- A non-empty subset *T* of a vector space *V* is called an affine subspace of *V*, when there exists a vector $\underline{v}_0 \in V$, and a subspace *S* of *V* such that $\forall\, \underline{t} \in T, \exists \underline{s} \in S, \underline{t} = \underline{s} + \underline{v}_0$
- Property
  - An affine subspace is a subspace of a vector space *V* only if it contains $\underline{0}$
- Note: An affine subspace generalize the concept of a plane in Euclidean geometry.
- Examples
  - $T_1 = \{ \underline{x} = a.\underline{x}_0 + \underline{y}_0 \mid \text{fixed } \underline{x}_0, \underline{y}_0 \in V, \forall a \in \mathbb{C} \}$, it is a subspace iff $\underline{y}_0 = \underline{0}$
  - $T_2 = \{ \underline{x} \in \mathbb{C}^{\mathbb{Z}} \mid x_n = 1, \forall n \neq 1,2,3 \}$, affine subspace of $\mathbb{C}^{\mathbb{Z}}$, it is not a subspace of $\mathbb{C}^{\mathbb{Z}}$ since the sequence of all "0" $\notin T_2$

Span/Linear independence: definition

- The span of a set of vectors S is the set of all finite linear combinations of vectors in S

$$S = \{\underline{\varphi}_k, k = 1, 2, \ldots, N\}$$

$$span(S) = span\{\underline{\varphi}_k\} = \left\{ \sum_{k=1}^{N} \alpha_k . \underline{\varphi}_k \middle| \alpha_k \in C, \underline{\varphi}_k \in S, N < \infty \right\}$$

- A set of vectors $S = \{\underline{\varphi}_k$, k=1, 2, …} is said linearly independent when the system of linear equations $\sum_k \alpha_k . \underline{\varphi}_k = \underline{0}$ admits as unique solution $\alpha_k = 0 \quad \forall k=1,2, \ldots$

Dimension: definition

- A vector space V is said to have dimension N when it contains a linear independent set of cardinality N and any other set of higher cardinality is linearly dependent. If no finite N exists, the vector space is infinite dimensional.

- Examples
  1. $\mathbb{R}^N$ has dimension N
  2. The vector space of polynomial functions of degree N has dimension N+1
  3. $\mathbb{C}^{\mathbb{Z}}$, $\mathbb{C}^{[a,b]}$ are infinite dimensional vector spaces

Inner product: definition

- An inner product on a vector space V over $\mathbb{C}$ (or $\mathbb{R}$) is a complex-(real-)valued function defined on VxV satisfying the following properties $\forall \underline{x},\underline{y},\underline{z} \in V$ and $\alpha \in \mathbb{C}$:

  a) $<\underline{x}+\underline{y},\underline{z}> = <\underline{x},\underline{z}> + <\underline{y},\underline{z}>$ linearity

  b) $<\alpha.\underline{x},\underline{y}> = \alpha.<\underline{x},\underline{y}>$ homogeneity

  [! $<\underline{x},\alpha.\underline{y}> = \alpha^*.<\underline{x},\underline{y}>$]

  c) $<\underline{x},\underline{y}> = <\underline{y},\underline{x}>^*$ [Hermitian symmetry]

  d) $<\underline{x},\underline{x}> \geq 0$ and $<\underline{x},\underline{x}> = 0$ iff $\underline{x} = \underline{0}$ [positive definiteness]

- Examples for $\forall (\underline{x},\underline{y}) \in \mathbb{C}^2$

  1. $<\underline{x},\underline{y}> = x_0 y_0^* + 5 x_1 y_1^*$ OK
  2. $<\underline{x},\underline{y}> = x_0^* y_0 + x_1^* y_1$ NO: violation of (b)
  3. $<\underline{x},\underline{y}> = x_0 y_0^*$ NO: violation of (d) ($<[0\ 1]^T,[0\ 1]^T> = 0$)

Inner product: standard definitions/orthogonality

- $\mathbb{C}^N$: $$\langle \underline{x},\underline{y} \rangle = \sum_{n=0}^{N-1} x_n . y_n^* \quad (\underline{x},\underline{y}) \in C^N$$
- Subspace of $\mathbb{C}^{\mathbb{Z}}$ leading to a convergent series $$\langle \underline{x},\underline{y} \rangle = \sum_{n \in Z} x_n . y_n^* \quad (\underline{x},\underline{y}) \in C^Z$$
- Subspace of $\mathbb{C}^{\mathbb{R}}$ leading to the existence of the integral $$\langle \underline{x},\underline{y} \rangle = \int_R x(t).y^*(t).dt \quad (\underline{x},\underline{y}) \in C^R$$

1. 2 vectors $\underline{x}$ and $\underline{y} \in V$ are said to be orthogonal ($\underline{x} \perp \underline{y}$) if $<\underline{x},\underline{y}> = 0$
2. A set of vectors S is said orthogonal whenever $\underline{x} \perp \underline{y}$ $\forall \underline{x},\underline{y} \in S$ with $\underline{x} \neq \underline{y}$

Inner product: orthogonality

3. A set of vectors S is said orthonormal whenever it is orthogonal and $\forall \underline{x} \in S$, $\langle \underline{x},\underline{x} \rangle = 1$
4. A vector $\underline{x}$ is said to be orthogonal to a set of vectors S ($\underline{x} \perp S$) when ($\underline{x} \perp \underline{s}$) $\forall \underline{s} \in S$
5. Two sets of vectors S_0 and S_1 are said to be orthogonal ($S_0 \perp S_1$) whenever $\underline{x}_0 \perp S_1$ $\forall \underline{x}_0 \in S_0$
6. Given a subspace S of a vector space V, the orthogonal complement of S, denoted $S^{\perp}$, is the set $\{\underline{x} \in V \mid \underline{x} \perp S\}$

- Properties
  - $S^{\perp}$ is a subspace of V
  - An orthonormal set $\{\underline{\varphi}_k\}$ is a linearly independent set

    (proof: expand $\underline{0}$ in $\langle \underline{0},\underline{\varphi}_i \rangle = 0$)

Inner product examples / Inner product space

- Example of an orthonormal set

  $\varphi_0(t) = 1 \qquad t \in [-½, ½]$

  $\varphi_k(t) = 2^{½} \cos(2k\pi t) \qquad t \in [-½, ½] \qquad k = 1, 2\ldots$

  $\{\varphi_k(t), k = 0, 1, \ldots\}$ is orthogonal to the set of odd functions S_{odd} defined over [-1/2,1/2]

- Definition

  A vector space equipped with an inner product is called an inner product space

Norm: definition

- A norm on a vector space V over $\mathbb{C}$ (or $\mathbb{R}$) is a real-valued function $||.||$ defined on V satisfying the following properties $\forall \underline{x},\underline{y} \in V$ and $\alpha \in \mathbb{C}$(or $\mathbb{R}$):

  a) $||\underline{x}|| \geq 0$ and $||\underline{x}|| = 0$ iff $\underline{x} = \underline{0}$ [positive definiteness]

  b) $||\alpha.\underline{x}|| = |\alpha|.||\underline{x}||$ [homogeneity]

  c) $||\underline{x}+\underline{y}|| \leq ||\underline{x}|| + ||\underline{y}||$ [triangle inequality]

  geometric interpretation: the length of any side of a triangle ≤ the sum of the lengths of the other two sides

- An inner product may be used to define a norm; in such a case the norm is said to be *induced* by the inner product

- Examples for $\forall\ \underline{x} \in \mathbb{C}^2$

  1. $||\underline{x}|| = (|x_0|^2 + 5|x_1|^2)^{1/2}$ OK
  2. $||\underline{x}|| = |x_0| + |x_1|$ OK
  3. $||\underline{x}|| = |x_0|$ NO: violation of (a)
  4. $||\underline{x}|| = \max(|x_0|, |x_1|)$ OK

Norm: standard definitions

- Euclidean norm in $\mathbb{C}^N$: $\|\underline{x}\|_2 = \sqrt{\sum_{n=0}^{N-1} |x_n|^2} \quad \underline{x} \in C^N$

- Euclidean norm in subspace of $\mathbb{C}^{\mathbb{Z}}$ for which the norm exists

$$\|\underline{x}\|_2 = \sqrt{\sum_{n\in Z} |x_n|^2} \quad \underline{x} \in C^Z$$

- Euclidean norm in subspace of $\mathbb{C}^{\mathbb{R}}$ for which the norm exists

$$\|\underline{x}\|_2 = \sqrt{\int_R |x(t)|^2 .dt} \quad \underline{x} \in C^R$$

- Or more generally

$$\|\underline{x}\|_2^2 = \langle \underline{x}, \underline{x} \rangle$$

Inner product induced norms: properties

- Pythagorean theorem

  $\forall \underline{x},\underline{y} \in V$, such that $\underline{x} \perp \underline{y}$ $\quad \|\underline{x}+\underline{y}\|^2 = \|\underline{x}\|^2 + \|\underline{y}\|^2$

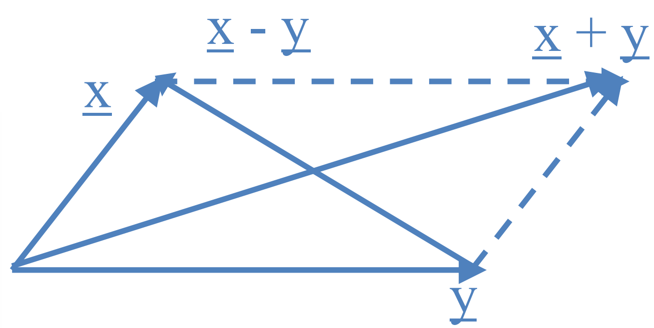
  proof: inner product properties + express $\langle \underline{x}+\underline{y}, \underline{x}+\underline{y} \rangle$

  more generally

  $\{\underline{x}_k\}_{k \in K}$ being an orthogonal set $\quad \left\| \sum_{k \in K} \underline{x}_k \right\|^2 = \sum_{k \in K} \|\underline{x}_k\|^2$

- Parallelogram law

  $\forall \underline{x},\underline{y} \in V \quad \|\underline{x}+\underline{y}\|^2 + \|\underline{x}-\underline{y}\|^2 = 2\left(\|\underline{x}\|^2 + \|\underline{y}\|^2\right)$



  – The parallelogram law is a necessary and sufficient condition for the norm to be induced by an inner product

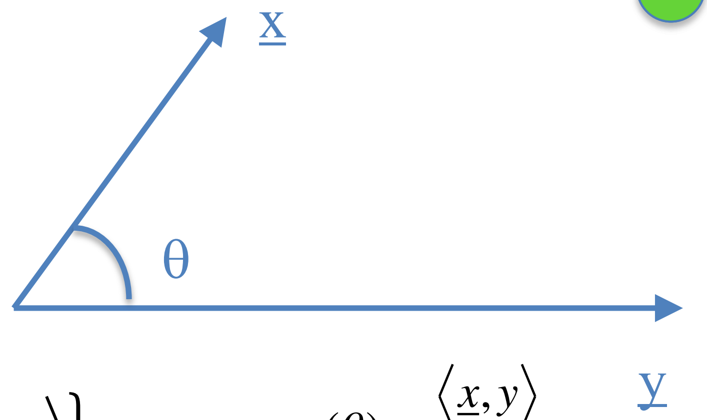
Inner product induced norms: properties

- Cauchy-Schwarz inequality

  $\forall \underline{x},\underline{y} \in V, \qquad \left|\langle \underline{x},\underline{y}\rangle\right| \le \|\underline{x}\|\cdot\|\underline{y}\|$

  – Equality holds when $\underline{x} = \alpha.\underline{y}$

  – proof: use non-negativity of $||k.\underline{x} + \underline{y}||^2$



$$\cos(\theta) = \frac{\langle \underline{x},\underline{y}\rangle}{\|\underline{x}\|\cdot\|\underline{y}\|}$$

$$0 \le \|k.\underline{x}+\underline{y}\|_2^2 = |k|^2\|\underline{x}\|_2^2 + \|\underline{y}\|_2^2 + 2.\mathrm{Re}\left\{\langle k.\underline{x},\underline{y}\rangle\right\}$$

$$= |k|^2\|\underline{x}\|_2^2 + \|\underline{y}\|_2^2 + 2.\mathrm{Re}\left\{k\langle \underline{x},\underline{y}\rangle\right\}$$

Choosing $k = -\langle\underline{x},\underline{y}\rangle^* / ||\underline{x}||^2$ leads to

$$\frac{\left|\langle \underline{x},\underline{y}\rangle^*\right|^2}{\|\underline{x}\|_2^2} + \|\underline{y}\|_2^2 + 2.\mathrm{Re}\left\{-\frac{\left|\langle \underline{x},\underline{y}\rangle\right|^2}{\|\underline{x}\|_2^2}\right\} \ge 0$$

$$\frac{\left|\langle \underline{x},\underline{y}\rangle\right|^2}{\|\underline{x}\|_2^2} \le \|\underline{y}\|_2^2$$

CVD

Normed vector space

- Normed vector space: A vector space equipped with a norm is called a normed vector space

- Note: caution is necessary to limit the subspace of V for which the norm exists

Metric: definition

- In a normed vector space V over $\mathbb{C}$ (or $\mathbb{R}$), the metric or distance between 2 vectors $\underline{x}$ and $\underline{y}$ is defined as the norm of the difference vector: $d(\underline{x},\underline{y}) = ||\underline{x}-\underline{y}||$
- Given a vector space V over $\mathbb{C}$(or $\mathbb{R}$), a distance may be more generally defined even in the absence of a norm as the real-valued function defined on VxV satisfying the following properties $\forall \underline{x},\underline{y},\underline{z} \in V$
  1. $d(\underline{x},\underline{y}) \geq 0$ [positivity]
  2. $d(\underline{x},\underline{y}) = 0 \Leftrightarrow \underline{x} = \underline{y}$
  3. $d(\underline{x},\underline{y}) = d(\underline{y}, \underline{x})$ [symmetric measure]
  4. $d(\underline{x},\underline{y}) \leq d(\underline{x},\underline{z}) + d(\underline{z},\underline{y})$ [triangular inequality]
- Example of a distance not induced by a norm in $\mathbb{R}$

$$d(x,y) = |\ \text{atan}(x) - \text{atan}(y)\ |$$

Standard spaces

- Standard inner product spaces
  - Space of complex-valued finite dimensional vectors: $\mathbb{C}^N$
  - Space of square-summable sequences: $l^2(\mathbb{Z}) \subset \mathbb{C}^{\mathbb{Z}}$ (infinite dimensional)
  - Space of square-integrable functions: $L^2(\mathbb{R}) \subset \mathbb{C}^{\mathbb{R}}$ (infinite dimensional)
  - Space of square-integrable functions over [a,b]: $L^2([a,b]) \subset \mathbb{C}^{[a,b]}$
  - Space of continuous functions $C[a,b] \subset \mathbb{C}^{[a,b]}$
  - Space of continuous functions with q continuous derivatives

    $C^q[a,b] \subset C^{q-1}[a,b] \subset \ldots \subset C^0[a,b]=C[a,b]$

    Note: $C^q[a,b]$ is not a complete space
  - Space of polynomial functions $\subset C^{\infty}[a,b]$
  - Space of random variables (RVs): inner product $\langle \underline{X},\underline{Y} \rangle = E[XY^*]$
    - The space of RVs with finite 2nd order moments is a normed vector space

Standard spaces

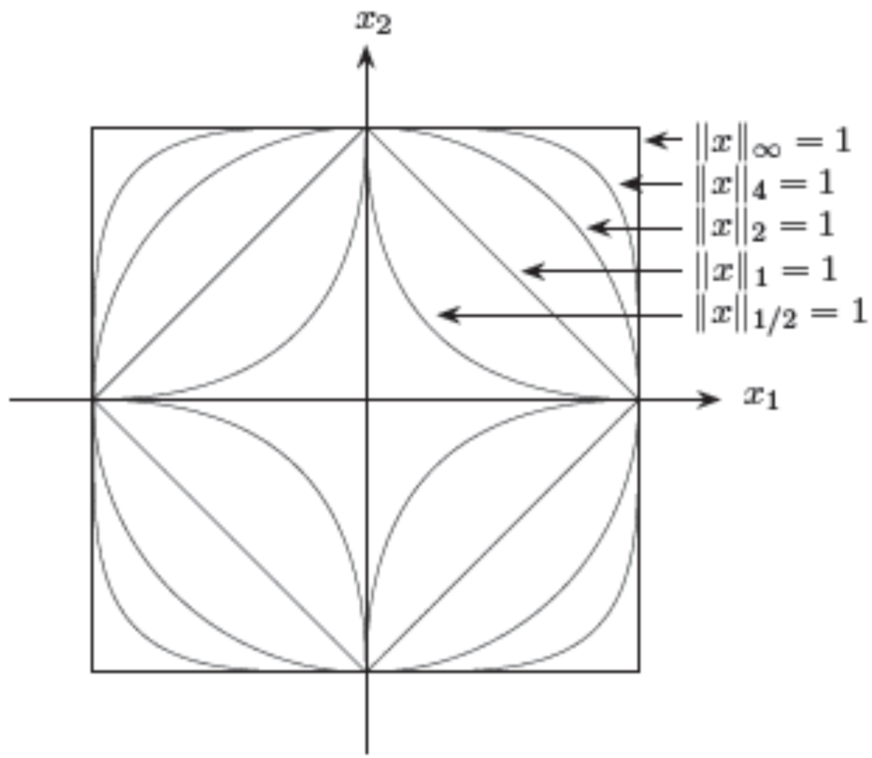
- Standard normed spaces
  - Space of complex-valued finite dimensional vectors: $\mathbb{C}^N$
    - p-norm: $$\|\underline{x}\|_p = \left(\sum_{n=0}^{N-1} |x_n|^p\right)^{1/p} \quad \underline{x} \in C^N$$
      - p=1: Manhattan norm
      - p=2: Euclidean norm
      - p=∞: $\|\underline{x}\|_\infty = \lim_{p\to\infty} \left(\sum_{n=0}^{N-1} |x_n|^p\right)^{1/p} = \max\left(|x_0|,|x_1|,\ldots,|x_{N-1}|\right) \quad \underline{x} \in C^N$
      - p∈[0,1) does not lead to a norm, but provides useful interpretation

$$\left\|\begin{pmatrix}1\\0\end{pmatrix}+\begin{pmatrix}0\\1\end{pmatrix}\right\|_{1/2} = (1+1)^2 = 4 > 2 = 1+1 = \left\|\begin{pmatrix}1\\0\end{pmatrix}\right\|_{1/2} + \left\|\begin{pmatrix}0\\1\end{pmatrix}\right\|_{1/2}$$

      - p=0, $||\underline{x}||_0$ accounts for the number of non zero components in $\underline{x}$
      - Any two norms bound each other within a constant factor
      - Only for p=2, the set of unit-norm vectors is invariant to a rotation of the coordinate system

l^p norms

- Set of unit-norm vectors in $\mathbb{R}^2$ for different l^p-measures



$l^p(\mathbb{Z})$ spaces

- l^p norm: $p \in [1,\infty)$

$$\|\underline{x}\|_p = \left(\sum_{n \in Z} |x_n|^p \right)^{1/p} \quad \underline{x} \in C^Z$$

  - l^∞ norm:

$$\|\underline{x}\|_\infty = \sup_{n \in Z} |x_n| \quad \underline{x} \in C^Z$$

  - for $p \in [1,\infty)$, the normed vector space $l^p(\mathbb{Z}) \subset \mathbb{C}^{\mathbb{Z}}$ correspond to the subspace formed by vectors in $\mathbb{C}^{\mathbb{Z}}$ with finite l^p norm

  - Property: $p<q \Rightarrow l^p(\mathbb{Z}) \subset l^q(\mathbb{Z})$
    - Corollary: If a sequence has finite l^1-norm, it has finite l^2-norm (the opposite is not necessarily true)
    - Example: $x_n=1/n$ $n=1,2,\dots$ and $x_n=0$ $n \leq 0$
      $||\underline{x}||_2 = \pi^2/6$ whereas $||\underline{x}||_1$ diverges

$\mathcal{L}^p(\mathbb{R})$ spaces

- $\mathcal{L}^p$ norm: p∈[1,∞)

$$\|\underline{x}\|_p = \left(\int_R |x(t)|^p \right)^{1/p} \qquad \underline{x} \in C^R$$

  - $\mathcal{L}^\infty$ norm: $\|\underline{x}\|_\infty = \operatorname*{ess\,sup}_{t \in R} |x(t)| \qquad \underline{x} \in C^R$
  - for p∈[1,∞), the normed vector space $\mathcal{L}^p(\mathbb{R}) \subset \mathbb{C}^{\mathbb{R}}$ corresponds to the subspace formed by vectors in $\mathbb{C}^{\mathbb{R}}$ with finite L^p norm

- Property: if p<q, $\mathcal{L}^p(\mathbb{R}) \subset \mathcal{L}^q(\mathbb{R})$
- It is possible to define similarly other $\mathcal{L}^p$ norm for other continuous time vector spaces such as $\mathbb{C}^{[a,b]}$

Convergence: definition

- Convergent sequence of vectors

A sequence of vectors $\underline{x}_0$, $\underline{x}_1$, … in a normed vector space V is said to converge to a vector $\underline{v} \in V$ when $\lim_{k\to\infty} ||\underline{v}-\underline{x}_k|| = 0$

  - In other words, given ε>0, $\exists K_\varepsilon$ such that $\|\underline{v} - \underline{x}_k\| < \varepsilon \quad \forall k > K_\varepsilon$
  - Note that the convergences may depend on the choice of the norm
    - Consider

$$x_k(t) = \begin{cases} 1 & t \in [0, k^{-1}] \\ 0 & otherwise \end{cases}$$

      - This sequence of vectors converges to v(t)=0 for all L^p norms with p<∞
      - It does not converge for the L^∞ norm

Closed subspace: definition

- A subspace S of a normed vector space V is said to be closed when it contains all limits of sequence of vectors in S.
- Properties
  - Subspaces of all finite-dimensional normed spaces are always closed
  - Span of infinite set of vectors may not be closed
  - The closure of a set is the set of all limit points of convergent sequences in the set
  - The closure of the span of an infinite set of vectors is the set of all convergent infinite linear combination. The closure of the span of a set of vectors is always a closed subspace

$$\overline{span}\left(\{\varphi_k\} k \in K\right) = \left\{ \sum_{k \in K} \alpha_k \varphi_k \middle| \; \alpha_k \in C \quad \text{and the sum converges} \right\}$$

Completeness / Hilbert spaces

- Cauchy sequence of vectors

  A sequence of vectors $\underline{x}_0$, $\underline{x}_1$, … in a normed vector space V is called a Cauchy sequence when given ε>0, $\exists K_\varepsilon$ such that

$$\left\| \underline{x}_k - \underline{x}_m \right\| < \varepsilon \quad \forall k,m > K_\varepsilon$$

– The elements of a Cauchy sequence stay arbitrarily close to each other.

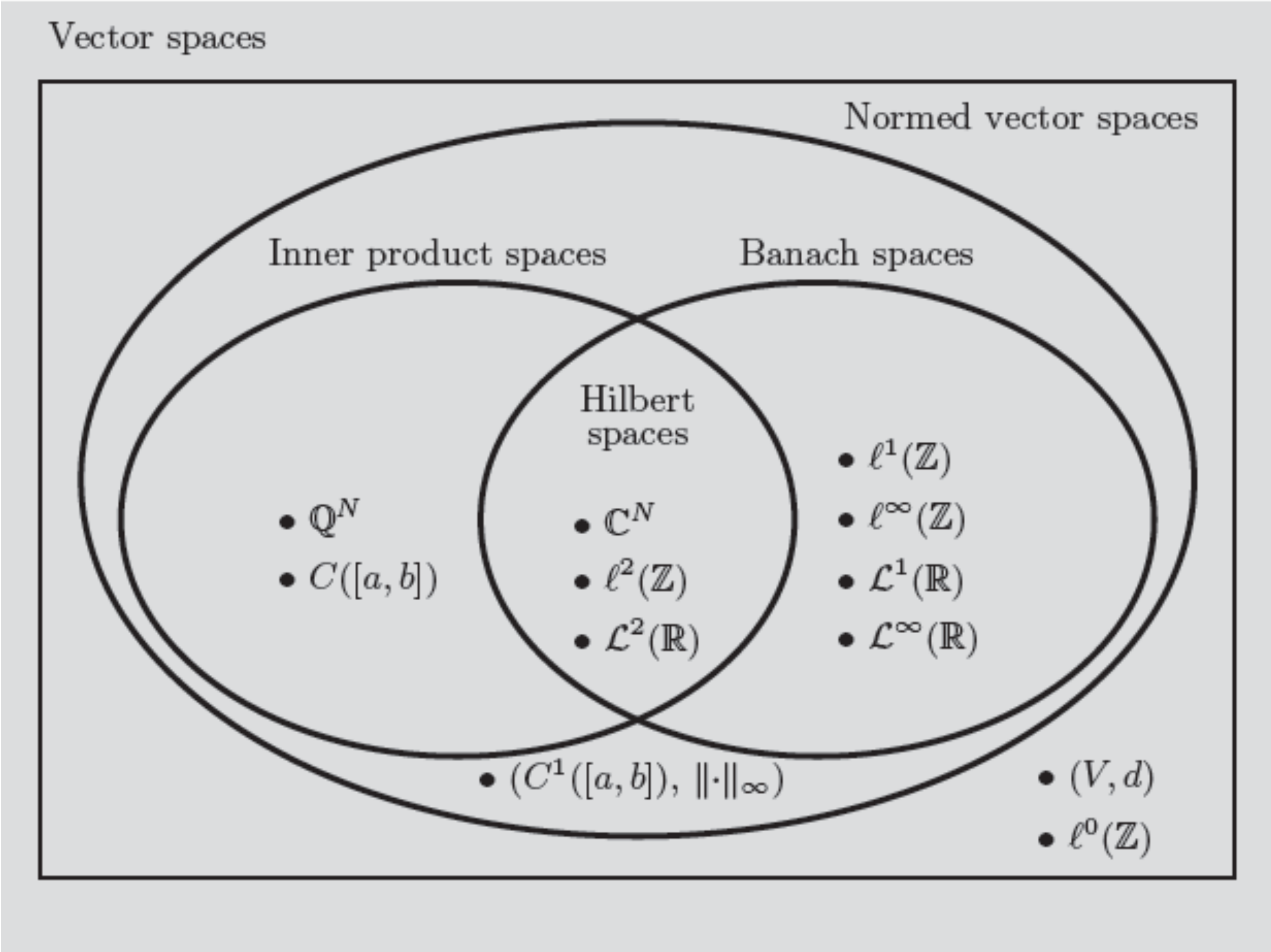
– For real-valued sequences, it must converge (but it may not be true for all normed vector spaces)

- A normed vector space V is said to be complete when every Cauchy sequence in V converges to a vector in V. A complete inner product space is called a Hilbert space.

Completeness / Banach spaces

- A complete normed vector space is called a Banach space
- Properties
  - $\mathbb{Q}$ is not a complete space since there are sequences in it converging to irrational numbers
  - All finite-dimensional spaces are complete
  - All $l^p(\mathbb{Z})$ spaces are complete; in particular $l^2(\mathbb{Z})$ is a Hilbert space
  - All $\mathcal{L}^p(\mathbb{R})$ spaces are complete; in particular $\mathcal{L}^2(\mathbb{R})$ is a Hilbert space $(p<\infty)$
  - $C^q([a,b])$ are not complete under the $\mathcal{L}^p$ norm for $p \in [0,\infty)$
  - The inner product space of random variables are complete and thus forms Hilbert space

Complete and non complete normed spaces



Relationship between different vector spaces. (V,d) is any vector space with a metric

Separability: definition

- A space is called separable when it contains a countable dense subset
- A Hilbert space contains a countable basis if and only if it is separable
  – A closed subspace of a separable Hilbert space is separable