EXERCISE LECTURE 29/09/2025

- EXAMPLE INFINITE SET OF VECTORS WHOSE SPAN IS NOT CLOSED

↳ in $\ell^2(\mathbb{Z})$ infinite sequences square-summable

Take $S = \{e_1, e_2, e_3, \dots\}$ where

$e_1 = (1, 0, \dots, 0, \dots, 0)$

$e_2 = (0, 1, 0, \dots, 0, \dots, 0)$

$e_3 = (0, 0, 1, 0, \dots, 0, \dots, 0)$

$\vdots$

$\underbrace{\qquad\qquad}_{\text{standard orth. basis}}$

the span(S)

$= \{(x_1, x_2, \dots) \in \ell^2 \mid$ only finitely many x_i's are non-zero $\}$

Consider the sequence

$v^{(1)} = (1, 0, \dots, 0, \dots)$

$v^{(2)} = (1, \frac{1}{2}, 0, \dots, 0, \dots)$

$v^{(3)} = (1, \frac{1}{2}, \frac{1}{3}, 0, \dots, 0, \dots)$

$\vdots$

$v^{(n)} = (1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, 0, \dots, 0, \dots)$

$\in \underline{\mathrm{Span}(S)}$

As $n \to \infty$

$v^{(n)} \to v = (1, \frac{1}{2}, \frac{1}{3}, \dots) \notin \mathrm{Span}(S)$

not finitely many non-zero

↳ $\in \ell^2(\mathbb{Z})$ $\quad \sum_{i=1}^{\infty} |v_i|^2 = \sum_{i=1}^{\infty} \frac{1}{i^2} = \pi^2/6 < \infty$

Therefore $v \in \overline{\mathrm{span}(S)}$ but $v \notin \mathrm{span}(S)$

- EXAMPLE SPACE THAT IS NOT COMPLETE

Space of continuous functions over $[0,1]$, i.e., $C[0,1]$ under the norm $\|\ \|_1$ (L_1-norm)

$$X = C[0,1] \qquad \|f\|_1 = \int_{-\infty}^{+\infty} |f(x)|\, dx$$

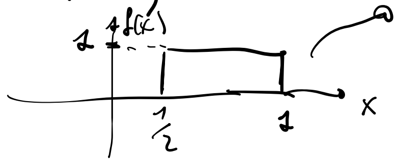
$(X, \|\cdot\|_1)$ is NOT COMPLETE

Take

$$f_n(x) = \begin{cases} 0 & x \le \frac{1}{2} - \frac{1}{n} \\ \frac{n}{2}\left(x - \frac{1}{2} + \frac{1}{n}\right) & \frac{1}{2} - \frac{1}{n} \le x \le \frac{1}{2} + \frac{1}{n} \\ 1 & x \geq \frac{1}{2} + \frac{1}{n} \end{cases}$$

$f_n \in C[0,1]$ (continue)

$f_n \xrightarrow{n \to +\infty} f$



function that is discontinuous

NOT COMPLETE

PROOF THAT $(\mathbb{R}^2, \|\cdot\|_2)$ IS COMPLETE

Given a couchy sequence $x^{(1)}, x^{(2)}, \dots, x^{(n)}, \dots$ of elements of $\mathbb{R}^2$

$x^{(n)} = (a_n, b_n) \quad a_n, b_n \in \mathbb{R}$

It is easy to see that

$$|a_n - a_m| < \|x^{(n)} - x^{(m)}\|_2 = \|(a_n, b_n) - (a_m, b_m)\|_2$$

$$\| (a_n - a_m, b_n - b_m) \|_2$$

$$|b_n - b_m| < \qquad = \sqrt{|a_n - a_m|^2 + |b_n - b_m|^2}$$

Ex - 1

2.20) CLOSED SUBSPACE AND $\ell^0(\mathbb{Z})$

$\ell^0(\mathbb{Z})$ set of complex-valued sequences with finite number of nonzero elements

(i) Show $\ell^0(\mathbb{Z}) \leq \ell^2(\mathbb{Z})$

(ii) Show $\ell^0(\mathbb{Z})$ is not a closed subspace of $\ell^2(\mathbb{Z})$

(i) $\forall v \in \ell^0(\mathbb{Z})$, $I = \{ i \mid v_i \neq 0 \}$, $|I| < \infty$

$|I| = n$ = # non-zero elements

$$\|v\|_2 = \Big(\sum_{i \in \mathbb{Z}} |v_i|^2\Big)^{\frac{1}{2}} = \Big(\sum_{i \in I} |v_i|^2\Big)^{\frac{1}{2}}$$

$$\leq \Big(n \max_{i \in I} |v_i|^2\Big)^{\frac{1}{2}} = \sqrt{n} \max_{i \in I} |v_i| < \infty$$

$$\Rightarrow v \in \ell^2(\mathbb{Z}) \xRightarrow{\forall v \in \ell^0(\mathbb{Z})} \ell^0(\mathbb{Z}) \subset \ell^2(\mathbb{Z})$$

(ii) Consider the sequence $v^{(n)} = [\cdots 0 \; \tfrac{1}{2} \; \tfrac{1}{3} \cdots \tfrac{1}{n} \, 0 \cdots] \in \ell^0(\mathbb{Z})$

$v = \lim_{n \to \infty} v^{(n)} \longrightarrow v \notin \ell^0(\mathbb{Z})$

However $v \in \ell^2(\mathbb{Z})$ since $\|v\|_2^2 = \sum_{i=1}^{\infty} \frac{1}{i^2} = \pi^2/6 < \infty$

$\Rightarrow \ell^2(\mathbb{Z})$ is not a closed subspace of $\ell^2(\mathbb{Z})$

Ex. 2

2.22) Complet even

let P be the inner product space of polynomials

$$\langle p, q\rangle = \int_0^1 p(t)\,\bar{q}(t)\,dt$$

and let

$$p_n(t) = \sum_{i=0}^{n} \frac{1}{2^i} t^i$$

sequence of polynomials

Prove $P \subset L^2([0,1)]$ is not Hilbert space.

Proof

Indicate $p(t) = \lim_{n\to\infty} p_n(t) = \lim_{n\to\infty} \sum_{i=1}^{n} \underbrace{\frac{1}{2^i} t^i}_{\left(\frac{t}{2}\right)^i}$ $\hookrightarrow \frac{t}{2} < 1$ $\boxed{t<2}$

we have that

$$p(t) = \frac{1}{1 - t/2} \qquad \boxed{0 \le t \le 1}$$

geometric series

$p_n(t) \in P$

$p(t) \notin P$ is not a polynomial

$\Rightarrow P$ is a inner product space but is not complete

$\Rightarrow P$ is not a Hilbert space

Ex. 3

2.24)

In a normed vector space every convergent sequence is a Cauchy sequence.

Proof.

Let (x_n) be a sequence convergent to x

$(x_n) \to x$, i.e.,

$$\lim_{n\to\infty} \|x - x_n\| = 0$$

Now we need to prove that (x_n) is Cauchy.

Then

$$\lim_{m,n\to\infty} \|x_n - x_m\| = \lim_{m,n\to\infty} \|x_n - x + x - x_m\|$$

Triangle inequality

$$\leq \lim_{m,n\to\infty} \underbrace{\|x_n - x\|}_{0} + \underbrace{\|x - x_m\|}_{0}$$

by convergence

$$= 0$$

Coordinate wise each component is $\boxed{\text{cauchy}}$.

Since $\mathbb{R}$ is complete $\exists\, a, b \in \mathbb{R}$:

$$a_n \xrightarrow{n\to+\infty} a \qquad b_n \xrightarrow{n\to+\infty} b$$

Finally

$$\| \overset{x^{(n)}}{(a_n, b_n)} - (a, b) \|_2 \longrightarrow 0$$

$$= \sqrt{(a_n - a)^2 + (b_n - b)^2} \leq \underset{\downarrow\, 0}{|a_n - a|} + \underset{\downarrow\, 0}{|b_n - b|}$$

$(\mathbb{R}^2, \|\cdot\|_2)$ IS COMPLETE

Exercise

2.5. *Linear independence*
Find the values of the parameter $a \in \mathbb{C}$ such that the following set is linearly independent:

$$U = \left\{ \begin{bmatrix} 0 & a^2 \\ 0 & j \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & a-1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ ja & 1 \end{bmatrix} \right\}.$$

For $a = j$, express the matrix $\begin{bmatrix} 0 & 5 \\ 2 & j-2 \end{bmatrix}$ as a linear combination of elements of U.

DEFINITION 2.5 (LINEARLY INDEPENDENT SET) The set of vectors $\{\varphi_0, \varphi_1, \ldots, \varphi_{N-1}\}$ is called *linearly independent* when $\sum_{k=0}^{N-1} \alpha_k \varphi_k = \mathbf{0}$ is true only if $\alpha_k = 0$ for all k. Otherwise, the set is linearly dependent. An infinite set of vectors is called linearly independent when every finite subset is linearly independent.

2.5)

$$\alpha_1 \begin{bmatrix} 0 & e^2 \\ 0 & j \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 & 1 \\ 1 & e-1 \end{bmatrix} + \alpha_3 \begin{bmatrix} 0 & 0 \\ je & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

if $\alpha_1 = \alpha_2 = \alpha_3 = 0$ only solution

$$\begin{bmatrix} 0 & \alpha_1 e^2 + \alpha_2 \\ \alpha_2 + \alpha_3 je & \alpha_1 j + \alpha_2(e-1) + \alpha_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{cases} \alpha_1 e^2 + \alpha_2 = 0 \\ \alpha_2 + \alpha_3 je = 0 \\ \alpha_1 j + \alpha_2 (e-1) + \alpha_3 = 0 \end{cases}$$

$(*)$ $\alpha_2 = -\alpha_1 e^2$

$\alpha_2 = -\alpha_3 je$

$$j e^2 \alpha_1 + e^2 \alpha_2 (e-1) + e^2 \alpha_3 = 0$$

$$-\alpha_2 j + e^2 \alpha_2 (e-1) + e^2 \alpha_3 = 0$$

$$\alpha_2 \left(e^2 (e-1) - j \right) + e^2 \alpha_3 = 0$$

$-\alpha_3 je$

$$\alpha_3 \left[je \left(e^2 (e-1) - j \right) - e^2 \right] = 0$$

$$\alpha_3 e \left[j \left(e^2 (e-1) - j \right) - e \right] = 0$$

$$\alpha_3 e \left[j \left(e^3 - e^2 - j \right) - e \right] = 0$$

$$\alpha_3 e \left[j (e^3 - e^2) + 1 - e \right] = 0$$

$$\begin{array}{c|ccc|c} & j & -j & -1 & 1 \\ 1 & & j & 0 & -1 \\ \hline & j & 0 & -1 & / \end{array}$$

$$\alpha_3 e (e-1) \underbrace{(je^2 - 1)}_{} = 0$$

$$(\sqrt{j}\, e - 1)(\sqrt{j}\, e + 1)$$

$$\sqrt{j} = \left(e^{i\pi/2} \right)^{1/2} = e^{i\pi/4}$$

$$= \cos(\pi/4) + j \sin(\pi/4)$$

$$= \frac{1}{\sqrt{2}} + j \frac{1}{\sqrt{2}} = \frac{1+j}{\sqrt{2}}$$

$a \notin \{0, 1, \frac{1}{\sqrt{j}}, -\frac{1}{\sqrt{j}}\}$ V is <u>independent</u>

$a = j$

$$\begin{bmatrix} 0 & 5 \\ 2 & j-2 \end{bmatrix} = \alpha_1 \begin{bmatrix} 0 & -1 \\ 0 & j \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 & 1 \\ 1 & j-1 \end{bmatrix} + \alpha_3 \begin{bmatrix} -0 & 0 \\ -1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -\alpha_1 + \alpha_2 \\ \alpha_2 - \alpha_3 & \alpha_1 j + \alpha_2 (j-1) + \alpha_3 \end{bmatrix} = \begin{bmatrix} 0 & 5 \\ 2 & j-2 \end{bmatrix}$$

$$\begin{cases} -\alpha_1 + \alpha_2 = 5 \Rightarrow \alpha_1 = \alpha_2 - 5 = -2 \\ \alpha_2 - \alpha_3 = 2 \Rightarrow \alpha_3 = \alpha_2 - 2 = 1 \\ \alpha_1 j + \alpha_2 (j-1) + \alpha_3 = j-2 \Longrightarrow (\alpha_2 - 5) j + \alpha_2 (j-1) + \alpha_2 - 2 = j - 2 \end{cases}$$

$$\alpha_2 j - 5j + \alpha_2 j \cancel{- \alpha_2} \cancel{+ \alpha_2} \cancel{-2} = j \cancel{-2}$$

$$2\alpha_2 \cancel{j} = 6 \cancel{j}$$

$$\Downarrow$$

$$\alpha_2 = 3$$

$[\alpha_1, \alpha_2, \alpha_3] = [-2, 3, 1]$

Exercise

2.7. *Inner product on* $\mathbb{C}^N$
Prove that $\langle x, y\rangle = y^* Ax$ is a valid inner product on $\mathbb{C}^N$ if and only if A is a Hermitian, positive definite matrix.

DEFINITION 2.7 (INNER PRODUCT) An *inner product* on a vector space V over $\mathbb{C}$ (or $\mathbb{R}$) is a complex-valued (or real-valued) function $\langle \cdot, \cdot\rangle$ defined on $V \times V$ with the following properties for any $x, y, z \in V$ and $\alpha \in \mathbb{C}$ (or $\mathbb{R}$):

(i) *Distributivity:* $\langle x + y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$.
(ii) *Linearity in the first argument:* $\langle \alpha x, y\rangle = \alpha\langle x, y\rangle$.
(iii) *Hermitian symmetry:* $\langle x, y\rangle^* = \langle y, x\rangle$.
(iv) *Positive definiteness:* $\langle x, x\rangle \geq 0$, and $\langle x, x\rangle = 0$ if and only if $x = \mathbf{0}$.

2.7) $A = A^H = (A^t)^* = (A^*)^t$

verify

(i) $\langle x+y, z\rangle = z^* A(x+y) = z^* A x + z^* A y$
$= \langle x, z\rangle + \langle y, z\rangle$ ✓

(ii) $\langle \alpha x, y\rangle = y^* A(\alpha x) = \alpha y^* A x = \alpha \langle x, y\rangle$ ✓

(iii) $\langle x, y\rangle^* = (y^* A x)^* = x^* A^* y = \langle y, x\rangle$ $A = A^*$ (hermitian)

(iv) $\langle x, x\rangle = x^* A x \geq 0$ definition of positive semidefinite matrix

Exercise

2.10. *Orthogonal transforms and ∞ norm*
Orthogonal transforms conserve the 2 norm, but not others, in general. Consider the ∞ norm (2.39b).

(i) Consider the set of real orthogonal transforms T_2 on $\mathbb{R}^2$, that is, plane rotations and rotoinversions (2.238). Give the best lower and upper bounds a_2 and b_2 so that

$$a_2 \;\leq\; \|T_2 x\|_\infty \;\leq\; b_2 \qquad \text{(P2.10-1)}$$

holds for all orthogonal T_2 and all vectors x of unit 2 norm.

(ii) Extend (P2.10-1) by giving the best lower and upper bounds a_N and b_N for the general case of real orthogonal transforms T_N on $\mathbb{R}^N$ with $N \geq 2$.

$$\|x\|_\infty \;=\; \max(|x_0|, |x_1|, \ldots, |x_{N-1}|).$$

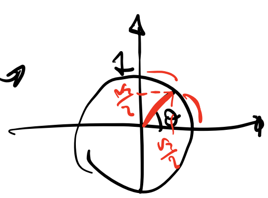
$\|x\|_\infty = \max(|x_0|, |x_1|, \ldots, |x_N|)$

$T_2 = \begin{bmatrix} \cos\theta & -\sin(\theta) \\ \sin\theta & \cos(\theta) \end{bmatrix}$ orthogonal rotation transform (i.e. preserves length and angles between vectors)

(i)

$a_2 \leq \|T_2 x\|_\infty \leq b_2$ holds for all orth. T_2 and all vectors x of unit 2 norm

$\|x\|_2 = 1$



max $\exists x$: $\|T_2 x\| = 1$

$\|T_2 x\|_\infty \leq 1 \quad \forall x \quad \forall T_2$

$\|T_2 x\|_\infty \geq \frac{\sqrt{2}}{2} \quad \forall x \; \forall T_2$

does not conserve $\|\cdot\|_\infty$

(ii) T_N on $\mathbb{R}^N$ $N \geq 2$ $a_N \leq \|T_N x\| \leq b_N$

$b_N = 1$

achieved when all components are equal $\Rightarrow$ diagonal of hypercube inscribed i.e. on hypersphere

Exercises

2.13. *Distances not necessarily induced by norms*

A *distance*, or *metric* $d : V \times V \to \mathbb{R}$ is a function with the following properties:

(i) *Nonnegativity:* $d(x, y) \geq 0$ for every x, y in V.

(ii) *Symmetry:* $d(x, y) = d(y, x)$ for every x, y in V.

(iii) *Triangle inequality:* $d(x, y) + d(y, z) \geq d(x, z)$ for every x, y, z in V.

(iv) *Identity of Indiscernibles:* $d(x, x) = 0$ and $d(x, y) = 0$ implies $x = y$.

The *discrete metric* is given by

$$d(x, y) = \begin{cases} 0, & \text{if } x = y; \\ 1, & \text{if } x \neq y. \end{cases}$$

Show that the discrete metric is a valid distance and is not induced by any norm.

(i) non negativity $d(x,y) \geq 0$ ✓

(ii) $\forall x,y \in V$ $\begin{cases} 0 = d(x,y) = d(y,x) & x = y \\ 1 = d(x,y) = d(y,x) & x \neq y \end{cases}$

(iii) suppose $x = y = z$ $0 \geq 0$ ok!

Suppose that they are not all equal

$d(x,y) + d(y,z) \geq 1 \geq d(x,z)$ ok!

(i) $d(x,x)=0$, $d(x,y)=0 \Rightarrow x=y$ ok!

Not induced by any norm

Consider $x = 2e_0 = [0 \dots 0\ 2\ 0 \dots 0]$
$y = -2e_1 = [0 \dots 0\ 0\ -2\ 0 \dots 0]$

For any $p \geq 1$

$$\|x-y\|_p = (2^p + 2^p)^{\frac{1}{p}} = (2^{p+1})^{\frac{1}{p}} = 2^{\frac{p+1}{p}} \geq 2 > 1$$

while $d(x,y) = 1$

$=$

$$\|x-y\| = d(x,y) \quad \forall x,y \in V$$

$\forall v \neq 0$ $\quad \|v\| = \|v-0\| = d(v,0) = 1$

hence, $\|\frac{1}{2}v\| = \frac{1}{2}\|v\| = \frac{1}{2}$

$$\|\tfrac{1}{2}v\| = \|\tfrac{1}{2}v - 0\| = d(\underbrace{\tfrac{1}{2}v}_{\neq 0}, 0) = 1$$

$1 = \frac{1}{2}$ contradiction

does not exist

Ex.

2.15) Show that ∞-norm is the natural extension of the p-norm

$$\lim_{p \to \infty} \|x\|_p = \max_{i=0,1,\dots,N-1} |x_i| \quad \forall x \in \mathbb{R}^N$$

Suppose $|x_0| \geq |x_1| \geq \cdots \geq |x_{N-1}|$

$$\lim_{p\to\infty} \|x\|_p = \lim_{p\to\infty} \left(|x_0|^p \Big(1 + \underbrace{\frac{|x_1|}{|x_0|}}_{\leq 1} + \cdots + \underbrace{\frac{|x_{N-1}|}{|x_0|}}_{\leq 1}\Big) \right)^{\frac{1}{p}}$$

$$|x_0| \leq \quad \leq \lim_{p\to\infty} \left(|x_0|^p N\right)^{\frac{1}{p}}$$

$$= \lim_{p\to\infty} |x_0| \left(N^{\frac{1}{p}}\right) \to 1$$

$$= |x_0|$$

2.16)

Quasi norm $p<1$

(i) Show that (iii) fails $p=\frac{1}{2}$

(ii) Show $x\in\mathbb{R}^N$, $\lim_{p\to 0} \|x\|_p^p$ gives # nonzero components in x.

(i) Take $x=\begin{bmatrix}1\\0\end{bmatrix}$ $y=\begin{bmatrix}0\\1\end{bmatrix}$

No TRIANGLE INEQ.

$$\|x+y\|_{\frac{1}{2}} = \left\|\begin{bmatrix}1\\1\end{bmatrix}\right\| = \left(1^{\frac{1}{2}}+1^{\frac{1}{2}}\right)^2 = 4 > 2 = 1+1 = \|x\|_{\frac{1}{2}} + \|y\|_{\frac{1}{2}}$$

(ii) $$\lim_{p\to 0} \|x\|_p^p = \lim_{p\to 0} \sum_{i=1}^{N} |x_i|^p = \sum_{i=1}^{N} \lim_{p\to 0} |x_i|^p$$

if $x_i \neq 0$
$\lim_{p\to 0} |x|^p = 1$

if $x_i = 0$
$\lim_{p\to 0} |x_i| = 0$

$x_i \neq 0$ contributes 1
$x_i = 0$ // 0