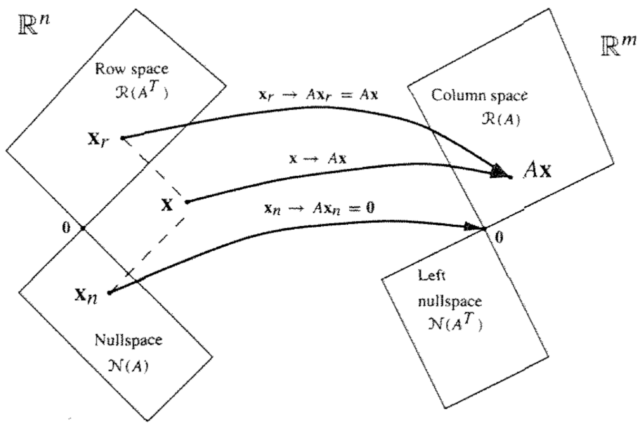
Exercise Lecture 27/10/2025

Matrix Operator Algebra

Matrix operator algebra



The four fundamental subspaces of a matrix operator

$\boxed{A \in \mathbb{K}^{n,m}}$ n rows , m columns

field

$A : \mathbb{K}^m \to \mathbb{K}^n$

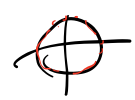
$y = Ax$

Boundness (matrix operator)

$A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \in \mathbb{R}^{2,2}$

$\underline{A} : \mathbb{R}^2 \to \mathbb{R}^2$

over $\mathbb{R}_2$ and with $\|\cdot\|_2$



$\|x\|_2 = 1$

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a generic element

$x = (\cos\theta , \sin\theta)$ $\theta \in \mathbb{R}$

$$\|\underline{\underline{A}}\| = \sup_{\theta} \left\| \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} \right\|_2$$

$$= \sup_{\theta} \left\| \begin{bmatrix} 3\cos\theta + \sin\theta \\ \cos\theta + 3\sin\theta \end{bmatrix} \right\|_2$$

$$= \sup_{\theta} \sqrt{(3\cos(\theta) + \sin(\theta))^2 + (\cos\theta + 3\sin\theta)^2}$$

$$= \sup_{\theta} \sqrt{9\underbrace{(\cos^2(\theta) + \sin^2(\theta))}_{=1} + \underbrace{\sin^2\theta + \cos^2\theta}_{=1} + \underbrace{12\sin(\theta)\cos(\theta)}_{6\sin(2\theta)}}$$

$$= \sup_{\theta} \sqrt{10 + 6\underbrace{\sin(2\theta)}_{-1 \le \cdot \le 1}} \overset{\theta = \frac{\pi}{4} + 2k\pi}{=} \sqrt{16} = \boxed{4}$$

$$\underline{\underline{\quad}}$$

$$\underline{\underline{A}} = \begin{bmatrix} 1 & j & 0 \\ 1 & 0 & j \end{bmatrix} \qquad A : \mathbb{C}^3 \to \mathbb{C}^2$$

$$\sup_{\|\underbrace{(x_1, x_2, x_3)}_{\underline{x}}\|_2 = 1} \|A\underline{x}\|_2 \to \sup_{\|x\|_2 = 1} \|A\underline{x}\|_2^2$$

$$\underline{\underline{\quad}}$$

$$A : \mathbb{C}^N \to \mathbb{C}^M \qquad A \in \mathbb{C}^{M,N}$$

Thm.

ℓ_2-gerated norm

$$\|A\|_2 = \sup_{\|x\|_2 = 1} \|Ax\|_2 = \sqrt{\lambda_{max}(A^H A)}$$

λ_{max} is the largest eigenvalue of A^*A

A^*A is the self-adjoint operator $(A^*A \in \mathbb{C}^{N,N})$

since $A^*A = (A^*A)^* \Rightarrow$ real eigenvalues and in the eigenvalue decomposition we have $A^*A = U \Lambda U^*$

→ eigenvectors

orthogonal (orthonormal) $\|x\|=1$

unitary matrix $UU^* = U^*U = I$

eigenvalue matrix

$$\sup_{\|x\|_2=1} \sqrt{\langle Ax, Ax\rangle} = \sup_{\|x\|_2=1} \sqrt{(Ax)^*(Ax)}$$

$$= \sup_{\|x\|_2=1} \sqrt{x^* A^*A\, x}$$

$$= \sup_{\|x\|_2=1} \sqrt{\underbrace{x^* U}_{(U^*x)^*} \Lambda U^* x}$$

$y = U^* x$

$\|y\|_2 = \|U^* x\|_2$

unitary

$$= \sup_{\|y\|_2=1} \sqrt{y^* \Lambda y} \rightarrow \mathbb{C}^{N,N}$$

$$= \sup_{\|y\|_2=1} \sqrt{\sum_{i=1}^{N} \lambda_i |y_i|^2}$$

all the weight to the eigenvalue of max value

$$= \sqrt{\lambda_{max}(A^*A)}$$

$$\sup_{\|x\|_2=1} \sqrt{\langle Ax, Ax\rangle} \overset{\text{adjoint}}{=} \sup_{\|x\|_2=1} \sqrt{\langle x, A^*Ax\rangle} \overset{\text{diagonalisable}}{=} \sup_{\|x\|_2=1} \sqrt{\langle x, \lambda_x x\rangle}$$

$$= \sup_{\|x\|_2=1} \sqrt{(\lambda_x x)^* x} = \sup_{\|x\|_2=1} \sqrt{\underbrace{\lambda_x^*}_{= \lambda_x \in \text{real}} x^* x} = \sup_{\|x\|_2=1} \sqrt{\lambda_x \underbrace{\|x\|_2}_{=1}}$$

$$= \sup_{\|x\|_2=1} \sqrt{\lambda_x} = \sqrt{\lambda_{max}(A^*A)}$$

take eigenvector associated to the maximum eigenvalue

$=$

$$A = \begin{bmatrix} 1 & j & 0 \\ 1 & 0 & j \end{bmatrix} \qquad \|A\|_2 = ?$$

$$A^*A = \begin{bmatrix} 1 & 1 \\ -j & 0 \\ 0 & -j \end{bmatrix} \begin{bmatrix} 1 & j & 0 \\ 1 & 0 & j \end{bmatrix} = \begin{bmatrix} 2 & j & j \\ -j & 1 & 0 \\ -j & 0 & 1 \end{bmatrix}$$

$$\det(A^*A - \lambda I) = 0 \longrightarrow \det \begin{bmatrix} 2-\lambda & j & j \\ -j & 1-\lambda & 0 \\ -j & 0 & 1-\lambda \end{bmatrix}$$

$$(2-\lambda)(1-\lambda)^2 - 2(1-\lambda) = 0$$

$$(1-\lambda)\Big[\underbrace{(2-\lambda)(1-\lambda)-2}_{\lambda^2 + \cancel{2} - 3\lambda - \cancel{2}}\Big] = 0$$

$$\lambda(1-\lambda)(\lambda-3) = 0 \qquad \lambda = 0, 1, 3$$

max

$$\|A\|_2 = \sqrt{3}$$

Find the maximizer, compute the eigenspace associated to $\underline{\lambda = 3}$.

$$\begin{bmatrix} -1 & j & j \\ -j & -2 & 0 \\ -j & 0 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{cases} -x + jy + jz = 0 \implies x = 2jy \\ \left.\begin{array}{l} -jx - 2y = 0 \\ -jx - 2z = 0 \end{array}\right\} \implies y = z \end{cases} \qquad V_3 = \langle (2j, 1, 1) \rangle$$

$x \in V_3$ and $\|x\|_2 = 1$

$$\|(2j,1,1)\|_2 = \sqrt{|2i|^2 + 1^2 + 1^2} = \sqrt{4+1+1} = \sqrt{6}$$

$$\boxed{x = (2i,1,1)/\sqrt{6} = \left(\sqrt{\tfrac{2}{3}}\, j, \tfrac{1}{\sqrt{6}}, \tfrac{1}{\sqrt{6}}\right)}$$

$$\left\| \begin{bmatrix} 1 & j & 0 \\ 1 & 0 & j \end{bmatrix} \begin{bmatrix} \sqrt{\frac{2}{3}}\, j \\ \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{bmatrix} \right\|_2 = \left\| \begin{bmatrix} \sqrt{\frac{2}{3}}\, j + \frac{j}{\sqrt{6}} \\ \sqrt{\frac{2}{3}}\, j + \frac{j}{\sqrt{6}} \end{bmatrix} \right\|_2$$

$$= \sqrt{2 \left| \sqrt{\tfrac{2}{3}}\, j + \tfrac{j}{\sqrt{6}} \right|^2} = \sqrt{2 \left| \tfrac{3}{\sqrt{6}}\, j \right|^2}$$

$$= \sqrt{\cancel{2} \cdot \frac{\cancel{9}\,3}{\cancel{6}\,\cancel{3}}} = \boxed{\sqrt{3}}$$