2.38. *Biorthogonal pair of bases of cosine functions*
Let $\Phi = \{\varphi_k\}_{k\in\mathbb{N}}$ be the set defined in Example 2.34 and $\Psi = \{\psi_k\}_{k\in\mathbb{N}}$ and $\widetilde{\Psi} = \{\widetilde{\psi}_k\}_{k\in\mathbb{N}}$ the sets defined in Example 2.40.

(i) Show that Ψ and $\widetilde{\Psi}$ satisfy the biorthogonality condition (2.110).

(ii) Show that $\overline{\text{span}}(\Psi) = \overline{\text{span}}(\widetilde{\Psi}) = \overline{\text{span}}(\Phi)$.

$$\Phi = \{\varphi_n\}_{n\in\mathbb{N}}$$

$$\Psi = \{\psi_n\}_{n\in\mathbb{N}}$$

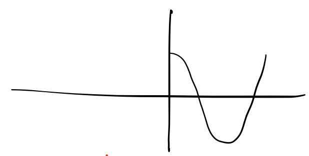
$$\boxed{\mathcal{L}^2([-\tfrac{1}{2},\tfrac{1}{2}])}$$

$$\psi_0(t) = 1$$

$$\psi_n(t) = \underbrace{\sqrt{2}\cos(2\pi n t)}_{\varphi_n(t)} + \frac{1}{2}\underbrace{\sqrt{2}\cos(2\pi(n+1)t)}_{\varphi_{n+1}(t)}$$

$$\widetilde{\psi}_n(t) = \sum_{m=1}^{n}\left(-\frac{1}{2}\right)^{n-m}\underbrace{\sqrt{2}\cos(2\pi m t)}_{\varphi_m(t)} \qquad \widetilde{\psi}_0(t) = 1$$

$\{\Psi, \widetilde{\Psi}\}$ - biorthogonal pair



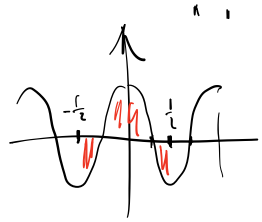
$\varphi_m(t) \to f_0 = m$
$T = \frac{1}{m}$

1°) $\langle \psi_0, \widetilde{\psi}_0 \rangle = 1$

$n \in \mathbb{N}\setminus\{0\}$

$\langle \psi_0, \widetilde{\psi}_n \rangle = 0 \quad \forall n = 1, 2, \dots$

coseni di frequenza $f_0 = n$
$T_0 = \frac{1}{n}$, $n \geq 1$



$\langle \psi_n, \underbrace{\widetilde{\psi}_0}_{=1} \rangle = 0 \quad \forall n = 1, 2, \dots \quad \mathbb{N}\setminus\{0\}$

$\langle \varphi_n, \tilde{\varphi}_\ell \rangle =$ $\quad n = 1, 2, \dots, \quad \ell = 1, 2, \dots$

$$\langle \varphi_n + \frac{1}{2}\varphi_{n+1}, \tilde{\varphi}_\ell \rangle = \langle \varphi_n + \frac{1}{2}\varphi_{n+1}, \sum_{m=1}^{\ell} \left(-\frac{1}{2}\right)^{\ell-m} \varphi_m \rangle$$

$$= \sum_{m=1}^{\ell} \left(-\frac{1}{2}\right)^{\ell-m} \langle \varphi_n + \frac{1}{2}\varphi_{n+1}, \varphi_m \rangle$$

$$= \sum_{m=1}^{\ell} \left(-\frac{1}{2}\right)^{\ell-m} \Big(\underbrace{\langle \varphi_n, \varphi_m \rangle}_{\delta_{n-m}} + \frac{1}{2} \underbrace{\langle \varphi_{n+1}, \varphi_m \rangle}_{\delta_{n+1-m}} \Big)$$

$$= \sum_{m=1}^{\ell} \left(-\frac{1}{2}\right)^{\ell-m} \Big(\overbrace{\delta_{n-m}}^{n=m} + \frac{1}{2} \overbrace{\delta_{n+1-m}}^{m=n+1} \Big)$$

$$= \left(-\frac{1}{2}\right)^{\ell-n} + \frac{1}{2}\left(-\frac{1}{2}\right)^{\ell-n-1}$$

$$= \left(-\frac{1}{2}\right)^{\ell-n} \left(1 + \frac{1}{2}\left(-\frac{1}{2}\right)^{-1} \right) = 0$$

$$\frac{1}{\cancel{2}} \cdot (-\cancel{2}) = -1$$

(ii) $\overline{\mathrm{span}}(\Psi) \subseteq \overline{\mathrm{span}}(\Phi)$

$\overline{\mathrm{span}}(\tilde{\Psi}) \subseteq \overline{\mathrm{span}}(\Phi)$

linear comb. of Φ el.

we need to prove

$\overline{\mathrm{span}}(\Phi) \subseteq \overline{\mathrm{span}}(\Psi)$

$\subseteq \overline{\mathrm{span}}(\tilde{\Psi})$

Since

$n>0$ $\qquad\qquad \varphi_0 = \psi_0 = \tilde{\psi}_0$

$$\varphi_n = \sum_{l=n}^{\infty}\left(-\frac{1}{2}\right)^{l-n}\overset{\varphi_l+\frac{1}{2}\varphi_{l+1}}{\tilde{\psi}_l} = \boxed{\tilde{\psi}_n - \frac{1}{2}\tilde{\psi}_{n-1}}$$

$$\sum_{m=1}^{n}\left(-\frac{1}{2}\right)^{k-m}\varphi_m - \sum_{m=1}^{k-1}\left(-\frac{1}{2}\right)^{k-m}\varphi_m$$

$$= \boxed{\varphi_n}$$

$$\varphi_n + \cancel{\frac{1}{2}\varphi_{n+1}} - \cancel{\frac{1}{2}\varphi_{n+1}} - \cancel{\frac{1}{4}\varphi_{n+2}} + \cancel{\frac{1}{4}\varphi_{n+2}} + \cancel{\frac{1}{8}\varphi_{n+3}} - \cdots$$

$$= \boxed{\varphi_n}$$

Exercises Lesson 19/11/2025

2.43. *Exploring the definition of frame*
Let $\Phi = \{\varphi_k\}_{k\in\mathcal{J}} \subset H$ be a frame.

(i) Show that if φ_j is a linear combination of $\{\varphi_k\}_{k\in\mathcal{J}\setminus\{j\}}$ for some $j\in\mathcal{J}$, the following property of a Riesz basis is *not* satisfied by $\{\varphi_k\}_{k\in\mathcal{J}}$: For any expansion $x = \sum_{k\in\mathcal{J}} \alpha_k\varphi_k$, condition (2.88) holds.

(ii) Show that for any $x\in H$, there exists an expansion $x = \sum_{k\in\mathcal{J}}\alpha_k\varphi_k$ such that condition (2.88) holds.

$$\varphi_j = \sum_{k\in J\setminus\{j\}} \beta_k \varphi_k \quad \text{per qualche } \beta\in\ell^2(J)$$

con $\beta_j = 0$ $\qquad \Phi = \{\varphi_k\}_{k\in J}$

$\forall x\in H$, visto che $\{\varphi_k\}$ è frame, $\exists \{\alpha_k\}_{k\in J}$, espansione

$$x = \sum_{k\in J}\alpha_k\varphi_k \quad \text{per qualche } \alpha\in\ell^2(J)$$

$\forall c\in\mathbb{R}$

$$x = \sum_{k\in J}\alpha_k\varphi_k = c\cdot 0 + \sum_{k\in J}\alpha_k\varphi_k$$

$$0 = c\left(\varphi_j - \sum_{k\in J\setminus\{j\}}\beta_k\varphi_k\right)$$

$$= \underbrace{c\varphi_j + \alpha_j\varphi_j}_{\varphi_j(c+\alpha_j)} + \sum_{k\in J\setminus\{j\}}(\alpha_k - c\beta_k)\varphi_k$$

non esiste $\exists$ max

La somma dei moduli quadri dei coeff. fra cui $|c+\alpha_j|^2$ è sicuramente unbounded per la scelta arb. di c.

(ii) Dual frame (canonical)

$$\tilde{\Phi} = (\Phi\Phi^*)^{-1}\Phi \quad (\text{prendo inverse}) \qquad \Phi\tilde{\Phi}^* = I$$

$$\tilde{\varphi}_n = (\Phi\Phi^*)^{-1}\varphi_n \quad n \in J$$

Sia Φ frame con costanti ottimali $\lambda_{min}, \lambda_{max}$

Un'espansione rispetto a Φ otteniamo $\tilde{\Phi}$

$$\alpha_n = \langle x, \tilde{\varphi}_n \rangle \quad n \in J$$

$$\sum_{n\in J} |\alpha_n|^2 = \sum_{n \in J} |\langle x, \tilde{\varphi}_n\rangle|^2 = \sum_{n\in J} |\langle x, (\Phi\Phi^*)^{-1}\varphi_n\rangle|^2$$

$$= \sum |\langle (\Phi\Phi^*)^{-1}x, \varphi_n\rangle|^2$$

self adjoint

$\Downarrow$ FRAME

$$\lambda_{min} \|(\Phi\Phi^*)^{-1}x\|^2 \leq \sum |\alpha_n|^2 \leq \lambda_{max} \|(\Phi\Phi^*)^{-1}x\|^2$$

$$\frac{1}{\lambda_{max}} I \leq (\Phi\Phi^*)^{-1} \leq \frac{1}{\lambda_{min}} I$$

$$\frac{1}{\lambda_{max}^2}\|x\|^2 \leq \|(\Phi\Phi^*)^{-1}x\|^2 \leq \frac{1}{\lambda_{min}^2}\|x\|^2$$

$\Downarrow$

$$\|x\|^2 \frac{\lambda_{min}}{\lambda_{max}^2} \leq \sum |\alpha_n|^2 \leq \frac{\lambda_{max}}{\lambda_{min}^2} \|x\|^2$$

2.45. *Dual frame*

Let $\Phi = \{\varphi_{k,n} = \delta_{n-k}\}_{k,n=0}^{3}$ be a basis for $\mathbb{R}^4$ and

$$\psi_{k,n} \;=\; \delta_{n-k} - \delta_{n-k-1}, \quad k = 0,\, 1,\, 2, \qquad \text{and} \qquad \psi_{3,n} = -\delta_{n-1} + \delta_{n-4}.$$

(i) Show that the $\Psi = \{\psi_k\}_{k=0}^{3}$ does not form a basis for $\mathbb{R}^4$. Which vectors from $\mathbb{R}^4$ are not in $\operatorname{span}(\{\psi_k\}_{k=0}^{3})$?

(ii) Show that $F = \{\varphi_k, \psi_k\}_{k=0}^{3}$ is a frame. Compute its frame bounds.

(iii) Find the canonical dual frame to F.

(iv) Let the Φ and Ψ be

$$\Phi \;=\; \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}, \qquad \Psi \;=\; \begin{bmatrix} 1 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}.$$

Comment on the completeness of Φ and Ψ in $\mathbb{R}^4$, show that $F = [\Phi \quad \Psi]$ is a frame, and find its frame bounds as well as the canonical dual frame.

$n=0,\dots,3$

(i)

$$\Phi = \{ \varphi_{k,n} = \delta_{n-k} \}_{k=0}^{3} = \{ \delta_n,\ \delta_{n-1},\ \delta_{n-2},\ \delta_{n-3} \}$$

$$\psi_{k,n} = \delta_{n-k} - \delta_{n-k-1} \quad k=0,1,2 \qquad \psi_{3,n} = \delta_{n-4} - \delta_n$$

$$\sum_{i=0}^{2} \psi_i = -\psi_3$$

$$\Phi = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \Psi = \begin{bmatrix} 1 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

$$\alpha(\quad) + \beta(\quad) + \gamma(\quad) + \delta(\quad)$$

$$\alpha\begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} + \beta\begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} + \gamma\begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix} + \cancel{\delta\begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}}$$

$$\begin{pmatrix} \alpha \\ \beta-\alpha \\ -\beta+\gamma \\ -\gamma \end{pmatrix}$$ la somma coeff. $= \boxed{0}$

$\forall\, x \in \mathbb{R}^4$ tale che la somma delle coeff. è $\neq 0$ non è nello span

(ii)

$$F = \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 1 \end{array}\right]$$

frame perché è full rank (sistema generatori)

λ_{min} e λ_{max} possono essere calcolati come λ_{min} e λ_{max} di FF^* ↘ $\lambda_{min} = 3$ e $\lambda_{max} = 5$

$$FF^* = \begin{bmatrix} 3 & -1 & 0 & -1 \\ -1 & 3 & -1 & 0 \\ 0 & -1 & 3 & -1 \\ -1 & 0 & -1 & 3 \end{bmatrix}$$

(iii) canonical dual frame

$\tilde{F} = (FF^*)^{-1} F \quad \dots \qquad \tilde{F} \in \mathbb{R}^{4,8}$

(iv) Né Ψ né Φ sono complete in $\mathbb{R}^4$ $\Rightarrow$ ma è un frame perché $F = [\Phi\ \Psi]$ è full-rank

↳ $\lambda_{min} = \lambda_{max} = 4$. Quindi F è tight $\boxed{\tilde{F} = \frac{1}{4} F}$

. *Tight frame with nonequal-norm vectors*
Assume $\alpha \in \mathbb{R}$, $\alpha \neq 0$, and let

$$\varphi_0 = \begin{bmatrix} 0 \\ \alpha \end{bmatrix}, \qquad \varphi_1 = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}, \qquad \varphi_2 = \begin{bmatrix} -\cos\theta \\ \sin\theta \end{bmatrix}.$$

For which values of θ and α is the above set a tight frame?

$$\alpha \neq 0$$

$$\Phi = \begin{bmatrix} 0 & \cos\theta & -\cos\theta \\ \alpha & \sin\theta & \sin\theta \end{bmatrix}$$

$$c \neq 0$$

$$\Phi \Phi^t = c\, I$$

$$\begin{bmatrix} 0 & \cos\theta & -\cos\theta \\ \alpha & \sin\theta & \sin\theta \end{bmatrix} \begin{bmatrix} 0 & \alpha \\ \cos\alpha & \sin\alpha \\ -\cos\theta & \sin\theta \end{bmatrix}$$

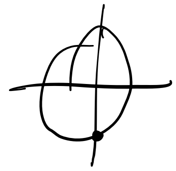
$$= \begin{bmatrix} 2\cos^2\theta & 0 \\ 0 & \alpha^2 + 2\sin^2\theta \end{bmatrix} = \begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix}$$

$$\alpha^2 + 2\big(\underbrace{\sin^2(\theta) - \cos^2(\theta)}_{-\cos(2\theta)}\big) \Rightarrow \alpha^2 = 2\cos(2\theta)$$

$$\Downarrow$$

$$|\alpha| = \sqrt{2\cos(2\theta)}$$

$$\boxed{0 < \cos(2\theta)} \leq 1$$



$$\cos(2\theta) > 0 \Rightarrow$$

$$-\frac{\pi}{2} + 2k\pi < 2\theta < \frac{\pi}{2} + 2k\pi$$

$$\begin{cases} -\frac{\pi}{4} + k\pi < \theta < \frac{\pi}{4} + k\pi \\ 0 < |\alpha| \leq \sqrt{2} \end{cases}$$