EXERCISES LESSON 10/12/2025

$\Delta t, \Delta f, \mu t, \mu f$

Calcola $\mu t, \Delta t, \mu f, \Delta f$ del segnale $x(t) = \text{tri}(t/T) = \begin{cases} 1 - |t|/T & |t| \le T \\ 0 & \text{altrimenti} \end{cases}$

$\mu t = 0$ perchè $x(t)$ segnale pari

$\|x(t)\|^2 = \frac{2}{3} T$ (energia del triangolo)

$$\Delta t^2 = \frac{1}{\|x\|^2} \int_{-\infty}^{+\infty} t^2 |x(t)|^2 dt = \frac{2}{\|x\|^2} \int_0^T t^2 \underbrace{\left(1 - \frac{t}{T}\right)^2}_{\left(1 + \frac{t^2}{T^2} - \frac{2t}{T}\right)} dt$$

$$= \frac{3}{T} \left[\frac{t^3}{3} + \frac{t^5}{5T^2} - \frac{2t^4}{4T} \right]_0^T$$

$$= \frac{3}{T} \left[\frac{T^3}{3} + \frac{T^3}{5} - \frac{T^3}{2} \right] = 3T^2 \left[\frac{10 + 6 - 15}{30} \right]$$

$$= \boxed{\frac{T^2}{10}}$$

$$\Delta t = \frac{T}{\sqrt{10}}$$

$$X(f) = T \,\text{sinc}^2(Tf)$$

$\mu f = 0$ ($x(t)$ segnale reale $\Rightarrow X(f)$ segnale pari)

$$\Delta f^2 = \frac{3}{T} \int_{-\infty}^{+\infty} f^2 T^2 \,\text{sinc}^4(Tf) \, df$$

$$x'(t) = \begin{cases} \frac{1}{T} & t \in [-T, 0] \\ -\frac{1}{T} & t \in [0, T] \\ 0 & \text{altrimenti} \end{cases} \qquad |x'(t)|^2 = \frac{1}{T^2} \quad -T \le t \le T$$

$$\mathcal{F}\{x'(t)\} = j2\pi f\, X(f)$$

parseval

$$\int_{-\infty}^{+\infty} |x'(t)|^2\, dt = \int_{-\infty}^{+\infty} |j2\pi f\, X(f)|^2\, df$$

$$= (2\pi)^2 \int_{-\infty}^{+\infty} f^2 |X(f)|^2\, df$$

$$= \int_{-T}^{T} \frac{1}{T^2}\, dt = 2\int_{0}^{T} \frac{1}{T^2}\, dt = \boxed{\frac{2}{T}}$$

$$\frac{1}{\|x\|^2} \frac{2}{T} \cdot \frac{1}{(2\pi)^2} = \left(\int_{-\infty}^{+\infty} f^2 |X(f)|^2\, df\right) \frac{1}{\|x\|^2}$$

$$\frac{3}{2T}$$

$$\frac{3}{\cancel{2}T} \cdot \frac{\cancel{2}}{T} \cdot \frac{1}{(2\pi)^2} = \Delta f^2 \Rightarrow \Delta f^2 = \frac{3}{T^2} \cdot \frac{1}{(2\pi)^2}$$

$$\Delta f = \frac{\sqrt{3}}{2\pi T}$$

$$\Delta t \cdot \Delta f = \frac{\cancel{T}}{\sqrt{10}} \cdot \frac{\sqrt{3}}{2\pi \cancel{T}} = \sqrt{\frac{3}{10}} \cdot \frac{1}{2\pi} > \boxed{\frac{1}{4\pi}}$$ Heisenberg

STFT

$$STFT(\tau, f) = \int_{-\infty}^{+\infty} x(t)\, w^*(t-\tau)\, e^{-j2\pi f t}\, dt$$

se $w(t) = rect(t/T)$

Supponiamo che $x(t) = e^{j2\pi f_0 t}$

calcoliamo

$$STFT(\tau, f) = \int_{-\infty}^{+\infty} \underbrace{e^{j2\pi f_0 t}\, rect\left(\frac{t-\tau}{T}\right)}_{x(t)w(t)} e^{-j2\pi f t}\, dt$$

$$\delta(f-f_0) * \left[T\, sinc(Tf)\, e^{-j2\pi f \tau}\right]$$

$$= T\, sinc(T(f-f_0)) \cdot e^{-j2\pi (f-f_0)\tau}$$

$$|STFT(\tau, f)| = T\, |sinc(T(f-f_0))|$$

ha max si ha $f = f_0 \Rightarrow \boxed{T}$

CWT

$x(t) = e^{j2\pi f_0 t}$, $f_0 > 0$

$\psi(t) = e^{-\frac{t^2}{2\sigma^2}}\, e^{j2\pi f_c t}$ $\quad f_c > 0$, $\sigma > 0$ (Morlet wavelet)

$$(a>0)$$

$$CWT_x(a,b) = \int_{-\infty}^{+\infty} x(t) \, \frac{1}{\sqrt{a}} \, \psi^*\left(\frac{t-b}{a}\right) dt$$

$$= \frac{1}{\sqrt{a}} \int_{-\infty}^{+\infty} e^{j2\pi f_0 t} \cdot e^{-\frac{t^2}{2\sigma^2}} \cdot e^{-j2\pi f_c t} \, dt$$

$$= \frac{1}{\sqrt{a}} \int_{-\infty}^{+\infty} e^{-\frac{\left(\frac{t-b}{a}\right)^2}{2\sigma^2}} \cdot e^{j2\pi f_0 t} \, e^{-j2\pi f_c \frac{(t-b)}{a}} \, dt$$