# Advanced Methods for Information Representation

Exercise Session 1 – Instructor Version with Solutions

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This sheet contains the three exercises actually covered in the first exercise session: linear independence, a metric not induced by a norm, and a non-closed subspace of $\ell^2(\mathbb{Z})$. The notation follows the convention used in *Foundations of Signal Processing*; in particular, $\mathrm{j} = \sqrt{-1}$.

## Exercise 1 – Linear independence with a parameter

Consider the set of matrices

$$U = \left\{ A_0 = \begin{bmatrix} 0 & a^2 \\ 0 & \mathrm{j} \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & 1 \\ 1 & a-1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0 \\ \mathrm{j}a & 1 \end{bmatrix} \right\}, \qquad a \in \mathbb{C}.$$

(i) Find all values of $a$ for which $U$ is linearly independent.

(ii) For $a = \mathrm{j}$, express

$$B = \begin{bmatrix} 0 & 5 \\ 2 & \mathrm{j}-2 \end{bmatrix}$$

as a linear combination of the elements of $U$.

### Solution

The set $U$ is linearly independent if and only if

$$\lambda_0 A_0 + \lambda_1 A_1 + \lambda_2 A_2 = 0$$

has only the trivial solution $\lambda_0 = \lambda_1 = \lambda_2 = 0$. Comparing the entries of the matrices gives

$$a^2\lambda_0 + \lambda_1 = 0, \tag{1}$$
$$\lambda_1 + \mathrm{j}a\lambda_2 = 0, \tag{2}$$
$$\mathrm{j}\lambda_0 + (a-1)\lambda_1 + \lambda_2 = 0. \tag{3}$$

From (1)–(2),

$$a^2\lambda_0 = -\lambda_1 = \mathrm{j}a\lambda_2.$$

Multiplying (3) by $a^2$ and substituting these relations yields

$$a(1 - \mathrm{j}a^2)(a-1)\lambda_2 = 0.$$

Now

$$1 - \mathrm{j}a^2 = 0 \quad \Longleftrightarrow \quad a^2 = -\mathrm{j} \quad \Longleftrightarrow \quad a = \pm\frac{1-\mathrm{j}}{\sqrt{2}}.$$

Therefore, if

$$a \notin \left\{0, 1, \frac{1-\mathrm{j}}{\sqrt{2}}, -\frac{1-\mathrm{j}}{\sqrt{2}}\right\},$$

then $\lambda_2 = 0$. Since $a \neq 0$, equations (1)–(2) then imply $\lambda_1 = \lambda_0 = 0$.

Conversely, for each of the four exceptional values the coefficient multiplying $\lambda_2$ above vanishes and the homogeneous system has a non-zero solution. Hence

$$\boxed{U \text{ is linearly independent } \iff a \notin \left\{0, 1, \pm\frac{1-\mathrm{j}}{\sqrt{2}}\right\}.}$$

For $a = \mathrm{j}$ we have

$$A_0 = \begin{bmatrix} 0 & -1 \\ 0 & \mathrm{j} \end{bmatrix}, \qquad A_1 = \begin{bmatrix} 0 & 1 \\ 1 & \mathrm{j}-1 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix}.$$

Solving

$$\lambda_0 A_0 + \lambda_1 A_1 + \lambda_2 A_2 = B$$

gives

$$\lambda_0 = -2, \qquad \lambda_1 = 3, \qquad \lambda_2 = 1.$$

Indeed,

$$-2A_0 + 3A_1 + A_2 = \begin{bmatrix} 0 & 5 \\ 2 & \mathrm{j}-2 \end{bmatrix} = B.$$

Thus

$$\boxed{B = -2A_0 + 3A_1 + A_2.}$$

**Blackboard takeaway.** Linear independence of structured objects such as matrices is tested exactly as for vectors: set a linear combination equal to zero and study uniqueness of the coefficients. The matrix entries simply turn the problem into a scalar linear system.

## Exercise 2 – A distance not induced by a norm

Let $V$ be a non-trivial vector space and define the *discrete metric*

$$d(x, y) = \begin{cases} 0, & x = y, \\ 1, & x \neq y. \end{cases}$$

(i) Prove that $d$ is a metric on $V$.

(ii) Prove that $d$ is not induced by any norm on $V$.

### Solution

We verify the axioms of a metric.

(i) **Non-negativity.** By definition, $d(x, y) \in \{0, 1\}$, so $d(x, y) \geq 0$.

(ii) **Identity of indiscernibles.** By definition,

$$d(x, y) = 0 \quad \iff \quad x = y.$$

(iii) **Symmetry.** The statements $x = y$ and $y = x$ are equivalent, hence

$$d(x, y) = d(y, x).$$

(iv) **Triangle inequality.** If $x = z$, then $d(x, z) = 0$ and the inequality is immediate. Suppose $x \neq z$. Then $d(x, z) = 1$. It is impossible to have simultaneously $x = y$ and $y = z$, because that would imply $x = z$. Therefore at least one of $d(x, y)$ and $d(y, z)$ equals 1, and hence

$$d(x, z) = 1 \leq d(x, y) + d(y, z).$$

Thus $d$ is a valid metric.

Now suppose, for contradiction, that $d$ were induced by a norm $\|\cdot\|$, so that

$$d(x, y) = \|x - y\| .$$

Choose any $x \neq 0$. Since $V$ is a vector space, $2x \neq 0$, and the discrete metric gives

$$d(0, x) = 1, \qquad d(0, 2x) = 1.$$

On the other hand, if the metric were norm-induced, homogeneity of the norm would imply

$$d(0, 2x) = \|2x\| = 2 \|x\| = 2d(0, x) = 2,$$

a contradiction. Therefore

| the discrete metric is not induced by any norm. |
|---|

**Blackboard takeaway.** Every norm-induced metric inherits linear structure. In particular, $d(0, \alpha x) = |\alpha| d(0, x)$. A general metric need not satisfy this scaling property. This is the quickest way to rule out norm-inducedness here.

## Exercise 3 – A subspace of $\ell^2(\mathbb{Z})$ that is not closed

Let

$$\ell^0(\mathbb{Z}) = \{x = (x_n)_{n \in \mathbb{Z}} : x_n \neq 0 \text{ for only finitely many } n\} .$$

Here $\ell^0(\mathbb{Z})$ denotes the set of finitely supported complex-valued sequences.

(i) Show that $\ell^0(\mathbb{Z})$ is a subspace of $\ell^2(\mathbb{Z})$.

(ii) Show that $\ell^0(\mathbb{Z})$ is not closed in $\ell^2(\mathbb{Z})$.

### Solution

**(i) $\ell^0(\mathbb{Z})$ is a subspace of $\ell^2(\mathbb{Z})$.** First, every finitely supported sequence belongs to $\ell^2(\mathbb{Z})$. Indeed, if

$$I = \{n \in \mathbb{Z} : x_n \neq 0\}$$

is finite, then

$$\|x\|_2^2 = \sum_{n \in \mathbb{Z}} |x_n|^2 = \sum_{n \in I} |x_n|^2 < \infty.$$

Thus $\ell^0(\mathbb{Z}) \subset \ell^2(\mathbb{Z})$.

The zero sequence has finite support. If $x, y \in \ell^0(\mathbb{Z})$, then

$$\operatorname{supp}(x+y) \subseteq \operatorname{supp}(x) \cup \operatorname{supp}(y),$$

which is a finite set. Hence $x+y \in \ell^0(\mathbb{Z})$. Finally, for any $\alpha \in \mathbb{C}$,

$$\operatorname{supp}(\alpha x) \subseteq \operatorname{supp}(x),$$

so $\alpha x \in \ell^0(\mathbb{Z})$. Therefore $\ell^0(\mathbb{Z})$ is a vector subspace of $\ell^2(\mathbb{Z})$.

**(ii) $\ell^0(\mathbb{Z})$ is not closed in $\ell^2(\mathbb{Z})$.** Consider the sequence $v = (v_n)_{n\in\mathbb{Z}}$ defined by

$$v_n = \begin{cases} 1/n, & n \geq 1, \\ 0, & n \leq 0. \end{cases}$$

Since

$$\|v\|_2^2 = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} < \infty,$$

we have $v \in \ell^2(\mathbb{Z})$. However, $v \notin \ell^0(\mathbb{Z})$ because it has infinitely many non-zero entries.

For $N \geq 1$, define the truncation

$$v_n^{(N)} = \begin{cases} 1/n, & 1 \leq n \leq N, \\ 0, & \text{otherwise.} \end{cases}$$

Each $v^{(N)}$ has finite support, hence $v^{(N)} \in \ell^0(\mathbb{Z})$. Moreover,

$$\left\|v - v^{(N)}\right\|_2^2 = \sum_{n=N+1}^{\infty} \frac{1}{n^2} \longrightarrow 0 \qquad (N \to \infty).$$

Therefore

$$v^{(N)} \longrightarrow v \qquad \text{in } \ell^2(\mathbb{Z}),$$

with every $v^{(N)} \in \ell^0(\mathbb{Z})$ but $v \notin \ell^0(\mathbb{Z})$. Hence $\ell^0(\mathbb{Z})$ does not contain all limits of convergent sequences of its elements, and therefore

$$\boxed{\ell^0(\mathbb{Z}) \text{ is not closed in } \ell^2(\mathbb{Z}).}$$

**Blackboard takeaway.** The key infinite-dimensional phenomenon is that finite-support sequences can approximate an infinite-support sequence arbitrarily well in the $\ell^2$ norm. In fact, $\ell^0(\mathbb{Z})$ is dense in $\ell^2(\mathbb{Z})$, even though it is a proper subspace.

## Summary of the three ideas

1. Linear independence reduces to uniqueness of the coefficients in a homogeneous system.
2. A metric may satisfy all metric axioms without respecting the homogeneity forced by a norm.
3. In infinite-dimensional normed spaces, a proper subspace need not be closed; truncation is a standard way to exhibit this phenomenon.