Advanced Methods for Information Representation
Exercise Session 2 – Exercises with Solutions

Inner products, norm geometry, ℓ^p and L^p spaces

University of Brescia

Notation

We follow the convention used in *Foundations of Signal Processing*: on $\mathbb{C}^N$,

$$\langle x, y \rangle = y^* x,$$

so the inner product is linear in its first argument and conjugate-linear in its second argument. For $1 \le p < \infty$,

$$\|x\|_p = \left(\sum_n |x_n|^p \right)^{1/p}, \qquad \|f\|_{L^p(E)} = \left(\int_E |f(t)|^p \, dt \right)^{1/p}.$$

1. Matrix-induced inner products

Exercise 1. *Let $A \in \mathbb{C}^{N \times N}$. Show that*

$$\langle x, y \rangle_A := y^* A x$$

defines an inner product on $\mathbb{C}^N$ if and only if A is Hermitian and positive definite.

Solution

Linearity in the first argument follows directly from matrix multiplication:

$$\langle \alpha x + \beta z, y \rangle_A = y^* A(\alpha x + \beta z) = \alpha \langle x, y \rangle_A + \beta \langle z, y \rangle_A .$$

For Hermitian symmetry we require

$$\langle x, y \rangle_A = \langle y, x \rangle_A^* .$$

But

$$\langle y, x \rangle_A^* = (x^* A y)^* = y^* A^* x.$$

Therefore Hermitian symmetry for every x, y is equivalent to

$$y^* A x = y^* A^* x \qquad \forall x, y,$$

which is equivalent to

$$A = A^*.$$

Finally, positive definiteness of the inner product is exactly

$$\langle x, x \rangle_A = x^* A x > 0 \qquad \forall x \neq 0.$$

This is precisely the definition of a positive definite matrix.
Hence

$$\boxed{\langle x, y\rangle_A = y^* A x \text{ is an inner product } \iff A = A^* \text{ and } A > 0.}$$

For example, if

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix},$$

then

$$\|x\|_A = \sqrt{x^* A x} = \sqrt{|x_1|^2 + 4|x_2|^2}.$$

2. Orthogonal transformations and the ℓ^∞ norm

Exercise 2. *Let T_N be a real orthogonal transformation on $\mathbb{R}^N$, and let $x \in \mathbb{R}^N$ satisfy $\|x\|_2 = 1$. Find the best constants a_N, b_N such that*

$$a_N \le \|T_N x\|_\infty \le b_N$$

for every orthogonal T_N and every Euclidean unit vector x.

Solution

Put

$$z = T_N x.$$

Since T_N is orthogonal,

$$\|z\|_2 = \|x\|_2 = 1.$$

Thus the problem is equivalent to finding the minimum and maximum of $\|z\|_\infty$ on the Euclidean unit sphere.
For the upper bound,

$$\|z\|_\infty \le \|z\|_2 = 1.$$

Equality is attained for $z = e_1$, so

$$b_N = 1.$$

For the lower bound, let $M = \|z\|_\infty$. Then $|z_k| \le M$ for every k, hence

$$1 = \|z\|_2^2 = \sum_{k=1}^{N} |z_k|^2 \le N M^2.$$

Therefore

$$M \ge \frac{1}{\sqrt{N}}.$$

Equality is attained when all components have the same modulus, for instance for

$$z = \frac{1}{\sqrt{N}}(1, \dots, 1)^T.$$

Consequently,

$$\boxed{\frac{1}{\sqrt{N}} \le \|T_N x\|_\infty \le 1.}$$

3. Triangle inequality and parallelogram law

Exercise 3. *Assume that*

$$\|x\| = \sqrt{\langle x, x\rangle}$$

is induced by an inner product.

(a) Use Cauchy–Schwarz to prove the triangle inequality.

(b) Prove the parallelogram law

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2 .$$

Solution

For the triangle inequality,

$$\begin{aligned}
\|x+y\|^2 &= \langle x+y, x+y\rangle \\
&= \|x\|^2 + \|y\|^2 + \langle x, y\rangle + \langle y, x\rangle \\
&= \|x\|^2 + \|y\|^2 + 2\,\mathrm{Re}\,\langle x, y\rangle \\
&\le \|x\|^2 + \|y\|^2 + 2\,|\langle x, y\rangle| \\
&\le \|x\|^2 + \|y\|^2 + 2\,\|x\|\,\|y\| \\
&= (\|x\| + \|y\|)^2 .
\end{aligned}$$

Taking square roots gives

$$\boxed{\|x+y\| \le \|x\| + \|y\| .}$$

For the parallelogram law,

$$\begin{aligned}
\|x+y\|^2 &= \|x\|^2 + \|y\|^2 + \langle x, y\rangle + \langle y, x\rangle , \\
\|x-y\|^2 &= \|x\|^2 + \|y\|^2 - \langle x, y\rangle - \langle y, x\rangle .
\end{aligned}$$

Adding the two identities cancels the cross terms:

$$\boxed{\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2 .}$$

4. Nesting of ℓ^p spaces

Exercise 4. *Let $1 \le p < q < \infty$. Prove that*

$$\ell^p(\mathbb{Z}) \subseteq \ell^q(\mathbb{Z}).$$

Give an explicit sequence in $\ell^2(\mathbb{Z})$ that is not in $\ell^1(\mathbb{Z})$.

Solution

Take $x \in \ell^p(\mathbb{Z})$. Then

$$\sum_{n\in\mathbb{Z}} |x_n|^p < \infty,$$

so necessarily $x_n \to 0$ as $|n| \to \infty$. Hence there is a finite set $F \subset \mathbb{Z}$ such that

$$|x_n| \le 1 \qquad (n \notin F).$$

Since $q > p$ and $0 \le |x_n| \le 1$ on the tail,

$$|x_n|^q \le |x_n|^p.$$

Therefore

$$\sum_{n \notin F} |x_n|^q \le \sum_{n \notin F} |x_n|^p < \infty.$$

The sum over the finite set F is finite, hence

$$\sum_{n \in \mathbb{Z}} |x_n|^q < \infty.$$

Thus

$$\boxed{\ell^p(\mathbb{Z}) \subseteq \ell^q(\mathbb{Z}), \qquad p < q.}$$

To see that the inclusion can be strict, define

$$x_n = \begin{cases} 1/n, & n \ge 1, \\ 0, & n \le 0. \end{cases}$$

Then

$$\sum_{n=1}^{\infty} \frac{1}{n^2} < \infty, \qquad \sum_{n=1}^{\infty} \frac{1}{n} = \infty.$$

Hence

$$\boxed{x \in \ell^2(\mathbb{Z}) \setminus \ell^1(\mathbb{Z}).}$$

Key point

For sequences, the tail is decisive. Once $|x_n| < 1$, increasing the exponent makes the terms smaller:

$$0 \le a \le 1, \quad p < q \quad \Longrightarrow \quad a^q \le a^p.$$

5. Hölder inequality: a short proof

Lemma (Hölder)**.** *Let* $1 < p, q < \infty$ *satisfy*

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Then

$$\int |f(t)g(t)|\, dt \le \|f\|_p \, \|g\|_q .$$

Solution

We use Young's inequality

$$ab \le \frac{a^p}{p} + \frac{b^q}{q}, \qquad a, b \ge 0,$$

which follows from the weighted arithmetic–geometric mean inequality.
Assume $\|f\|_p , \|g\|_q > 0$ and define

$$F(t) = \frac{|f(t)|}{\|f\|_p}, \qquad G(t) = \frac{|g(t)|}{\|g\|_q}.$$

Then

$$\int F^p = 1, \qquad \int G^q = 1.$$

Applying Young pointwise,

$$F(t)G(t) \le \frac{F(t)^p}{p} + \frac{G(t)^q}{q}.$$

Integrating,

$$\int FG \le \frac{1}{p}\int F^p + \frac{1}{q}\int G^q = \frac{1}{p} + \frac{1}{q} = 1.$$

Multiplying by $\|f\|_p \|g\|_q$ yields

$$\boxed{\int |fg| \le \|f\|_p \|g\|_q .}$$

The cases $\|f\|_p = 0$ or $\|g\|_q = 0$ are immediate.

6. Finite-measure L^p spaces: reverse inclusion

Exercise 5. *Let $1 \le p < q < \infty$ and let $[a, b]$ be a finite interval.*

(a) Prove that

$$L^q([a,b]) \subseteq L^p([a,b]).$$

More precisely, prove

$$\boxed{\|f\|_{L^p([a,b])} \le (b-a)^{\frac{1}{p}-\frac{1}{q}} \|f\|_{L^q([a,b])} .}$$

(b) Show that the reverse inclusion is false in general.

Solution

For part (a), write

$$\int_a^b |f(t)|^p \, dt = \int_a^b |f(t)|^p \cdot 1 \, dt.$$

Apply Hölder with conjugate exponents

$$r = \frac{q}{p}, \qquad r' = \frac{q}{q-p}.$$

Then

$$\begin{aligned}\int_a^b |f|^p &\le \left(\int_a^b |f|^{pr}\right)^{1/r} \left(\int_a^b 1^{r'}\right)^{1/r'} \\ &= \left(\int_a^b |f|^q\right)^{p/q} (b-a)^{1-p/q}.\end{aligned}$$

Taking the pth root gives

$$\boxed{\|f\|_p \le (b-a)^{1/p-1/q} \|f\|_q .}$$

Therefore

$$\boxed{L^q([a,b]) \subseteq L^p([a,b]).}$$

For part (b), work on $(0,1)$ and choose

$$\frac{1}{q} < \alpha < \frac{1}{p}.$$

For example,

$$\alpha = \frac{1}{2}\left(\frac{1}{p} + \frac{1}{q}\right).$$

Set

$$f(t) = t^{-\alpha}.$$

For any $r > 0$,

$$\int_0^1 |f(t)|^r \, dt = \int_0^1 t^{-\alpha r} \, dt,$$

which is finite if and only if

$$\alpha r < 1.$$

Since $\alpha p < 1$ but $\alpha q > 1$,

$$f \in L^p(0,1), \qquad f \notin L^q(0,1).$$

Thus the reverse inclusion fails in general.

7. On $\mathbb{R}$, L^p and L^q are incomparable

Exercise 6. *Let $1 \le p < q < \infty$. Show that, in general,*

$$L^p(\mathbb{R}) \not\subseteq L^q(\mathbb{R}) \qquad \textit{and} \qquad L^q(\mathbb{R}) \not\subseteq L^p(\mathbb{R}).$$

Solution

Choose

$$\alpha = \frac{1}{2}\left(\frac{1}{p} + \frac{1}{q}\right),$$

so that

$$\frac{1}{q} < \alpha < \frac{1}{p}.$$

First define

$$f(t) = t^{-\alpha}\mathbf{1}_{(0,1)}(t).$$

Then

$$\int_0^1 |f(t)|^r \, dt = \int_0^1 t^{-\alpha r} \, dt$$

is finite exactly when $\alpha r < 1$. Hence

$$\alpha p < 1 \Rightarrow f \in L^p(\mathbb{R}), \qquad \alpha q > 1 \Rightarrow f \notin L^q(\mathbb{R}).$$

Thus

$$L^p(\mathbb{R}) \not\subseteq L^q(\mathbb{R}).$$

Now define

$$g(t) = t^{-\alpha}\mathbf{1}_{(1,\infty)}(t).$$

Then

$$\int_1^\infty |g(t)|^r \, dt = \int_1^\infty t^{-\alpha r} \, dt$$

is finite exactly when $\alpha r > 1$. Hence

$$\alpha q > 1 \Rightarrow g \in L^q(\mathbb{R}), \qquad \alpha p < 1 \Rightarrow g \notin L^p(\mathbb{R}).$$

Thus

$$L^q(\mathbb{R}) \not\subseteq L^p(\mathbb{R}).$$

Therefore neither space contains the other in general.

Key point

The three situations are worth memorizing:

$\ell^p(\mathbb{Z})$	:	$p < q \implies \ell^p \subseteq \ell^q$,
$L^p([a,b])$	:	$p < q \implies L^q \subseteq L^p$,
$L^p(\mathbb{R})$	:	in general, no inclusion in either direction.

For sequences, the behavior of a tail of small numbers dominates. On a finite interval, larger exponents penalize large peaks more strongly. On $\mathbb{R}$, both local singularities and behavior at infinity may matter.

8. Convergent sequences are Cauchy

Exercise 7. *Show that every convergent sequence in a normed vector space is a Cauchy sequence.*

Solution

Assume $x_n \to x$. Given $\varepsilon > 0$, there exists N such that

$$\|x_n - x\| < \frac{\varepsilon}{2} \qquad \forall n \geq N.$$

Therefore, for $n, m \geq N$,

$$\begin{aligned} \|x_n - x_m\| &= \|x_n - x + x - x_m\| \\ &\leq \|x_n - x\| + \|x - x_m\| \\ &< \varepsilon. \end{aligned}$$

Hence (x_n) is Cauchy.

Backup exercises

B1. The limit $p \to \infty$

Exercise 8. *For $x \in \mathbb{R}^N$, prove*

$$\lim_{p\to\infty} \|x\|_p = \|x\|_\infty .$$

Solution

Let

$$M := \|x\|_\infty = \max_{0\le i\le N-1} |x_i|.$$

If $M = 0$, then $x = 0$ and the statement is immediate. Assume $M > 0$ and set

$$y = \frac{x}{M}.$$

Then $\|y\|_\infty = 1$: every coordinate satisfies $|y_i| \le 1$ and at least one coordinate has modulus exactly 1. Thus

$$1 \le \sum_{i=0}^{N-1} |y_i|^p \le N.$$

Taking pth roots,

$$1 \le \|y\|_p \le N^{1/p}.$$

Since $N^{1/p} \to 1$, the squeeze theorem gives

$$\|y\|_p \to 1.$$

By homogeneity,

$$\|x\|_p = M\,\|y\|_p \to M = \|x\|_\infty .$$

Hence

$$\boxed{\lim_{p\to\infty} \|x\|_p = \|x\|_\infty .}$$

B2. Quasinorms and the ℓ^0 quantity

Exercise 9. *For $p = 1/2$, show that*

$$\|x\|_{1/2} = \left(\sum_i |x_i|^{1/2}\right)^2$$

fails the triangle inequality. Then show that for $x \in \mathbb{R}^N$,

$$\lim_{p\downarrow 0} \|x\|_p^p$$

equals the number of nonzero components of x.

Solution

Take

$$x = (1, 0), \qquad y = (0, 1).$$

Then

$$\|x\|_{1/2} = 1, \qquad \|y\|_{1/2} = 1,$$

while

$$\|x + y\|_{1/2} = (1 + 1)^2 = 4.$$

Therefore

$$4 > 2 = \|x\|_{1/2} + \|y\|_{1/2},$$

so the triangle inequality fails.

For the second part,

$$\|x\|_p^p = \sum_{i=0}^{N-1} |x_i|^p.$$

Since the sum is finite,

$$\lim_{p\downarrow 0} \|x\|_p^p = \sum_{i=0}^{N-1} \lim_{p\downarrow 0} |x_i|^p.$$

For each coordinate,

$$\lim_{p\downarrow 0} |x_i|^p = \begin{cases} 1, & x_i \neq 0, \\ 0, & x_i = 0. \end{cases}$$

Thus

$$\boxed{\lim_{p\downarrow 0} \|x\|_p^p = \#\{i : x_i \neq 0\}.}$$

This quantity is often denoted by $\|x\|_0$, although it is not a norm.

B3. An incomplete inner-product space

Exercise 10. *Let $\mathcal{P}$ be the space of polynomials on $[0,1]$ equipped with*

$$\langle p, q\rangle = \int_0^1 p(t)q(t)^*\, dt.$$

Consider

$$p_k(t) = \sum_{i=0}^{k} \frac{t^i}{2^i}.$$

Show that (p_k) is Cauchy but its $L^2([0,1])$ limit is not a polynomial. Conclude that $\mathcal{P}$ is not complete and hence is not a Hilbert space.

Solution

Let $m > k$. For $t \in [0,1]$,

$$\begin{aligned} |p_m(t) - p_k(t)| &= \left| \sum_{i=k+1}^{m} \left(\frac{t}{2}\right)^i \right| \\ &\leq \sum_{i=k+1}^{\infty} \left(\frac{1}{2}\right)^i = 2^{-k}. \end{aligned}$$

Hence

$$\|p_m - p_k\|_2^2 = \int_0^1 |p_m(t) - p_k(t)|^2\, dt \leq 2^{-2k},$$

so

$$\|p_m - p_k\|_2 \leq 2^{-k} \to 0.$$

Therefore (p_k) is Cauchy.
For each $t \in [0,1]$, the geometric series gives

$$p(t) = \sum_{i=0}^{\infty} \left(\frac{t}{2}\right)^i = \frac{1}{1 - t/2}.$$

Moreover,

$$|p(t) - p_k(t)| \leq 2^{-k},$$

so

$$\|p - p_k\|_2 \leq 2^{-k} \to 0.$$

Thus $p_k \to p$ in $L^2([0,1])$.
However,

$$p(t) = \frac{1}{1 - t/2}$$

is not a polynomial. Hence (p_k) is a Cauchy sequence in $\mathcal{P}$ whose limit does not belong to $\mathcal{P}$. Therefore $\mathcal{P}$ is not complete, and consequently it is not a Hilbert space.