Advanced Methods for Information Representation
Exercise Session 4 – Solutions

Orthogonal projections, bases, Bessel, Parseval and Riesz bases

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This document contains the complete solutions to the exercises discussed in Exercise Session 4. The notation follows *Foundations of Signal Processing*.

Useful facts and notation

An orthonormal set $\{\varphi_k\}_{k\in K}$ satisfies

$$\langle \varphi_k, \varphi_j \rangle = \delta_{kj}.$$

It is an orthonormal basis of a Hilbert space H when

$$\overline{\operatorname{span}\{\varphi_k : k \in K\}} = H.$$

For a finite subset $F \subset K$, define the partial orthogonal projection

$$P_F x = \sum_{k\in F} \langle x, \varphi_k \rangle \, \varphi_k.$$

For a sequence $\{\varphi_k\}$, a convenient definition of a Riesz basis is: there exists a bounded invertible operator T and an orthonormal basis $\{e_k\}$ such that

$$\varphi_k = T e_k.$$

Equivalently, there exist constants $0 < A \le B < \infty$ such that for every finitely supported scalar sequence c,

$$A \sum_k |c_k|^2 \le \left\| \sum_k c_k \varphi_k \right\|^2 \le B \sum_k |c_k|^2.$$

1. Self-adjoint projection ⇒ orthogonal projection

Exercise 1. *Let $P : H \to H$ be a bounded linear operator on a Hilbert space such that*

$$P^2 = P.$$

Assume also that

$$P^* = P.$$

(a) Show that for every $x \in H$,

$$x - Px \in \operatorname{Ker}(P).$$

(b) Prove that

$$\operatorname{Ker}(P) = \operatorname{Ran}(P)^{\perp}.$$

(c) Conclude that P is the orthogonal projection onto $\operatorname{Ran}(P)$.

Solution

Because $P^2 = P$,

$$P(x - Px) = Px - P^2x = 0,$$

so

$$\boxed{x - Px \in \mathrm{Ker}(P).}$$

Now let $z \in \mathrm{Ker}(P)$ and $y \in \mathrm{Ran}(P)$. Write $y = Pw$ for some $w \in H$. Then

$$\langle z, y\rangle = \langle z, Pw\rangle = \langle P^*z, w\rangle = \langle Pz, w\rangle = 0.$$

Hence

$$\mathrm{Ker}(P) \subseteq \mathrm{Ran}(P)^\perp.$$

For the reverse inclusion, let $z \perp \mathrm{Ran}(P)$. Since $Pz \in \mathrm{Ran}(P)$,

$$0 = \langle z, Pz\rangle = \langle P^*z, z\rangle = \langle Pz, z\rangle .$$

A cleaner route uses $z \perp \mathrm{Ran}(P)$ and the decomposition

$$z = Pz + (z - Pz),$$

where $Pz \in \mathrm{Ran}(P)$ and $z - Pz \in \mathrm{Ker}(P) \subseteq \mathrm{Ran}(P)^\perp$. Thus Pz is orthogonal both to z and to $z - Pz$, so

$$0 = \langle z, Pz\rangle = \langle Pz, Pz\rangle + \langle z - Pz, Pz\rangle = \|Pz\|^2 .$$

Therefore $Pz = 0$ and $z \in \mathrm{Ker}(P)$. Hence

$$\boxed{\mathrm{Ker}(P) = \mathrm{Ran}(P)^\perp.}$$

Finally, for every x,

$$\boxed{x = Px + (x - Px)},$$

with

$$Px \in \mathrm{Ran}(P), \qquad x - Px \in \mathrm{Ran}(P)^\perp.$$

This is exactly the orthogonal decomposition of x relative to $\mathrm{Ran}(P)$. Therefore

$$\boxed{P \text{ is the orthogonal projector onto } \mathrm{Ran}(P).}$$

Conceptual message

Idempotency $P^2 = P$ determines the target subspace. Self-adjointness forces the complementary direction to be orthogonal to that subspace.

2. Bessel inequality

Exercise 2 (Bessel inequality)**.** *Let $\{\varphi_k\}_{k\in K}$ be an orthonormal set in a Hilbert space H, and let $x \in H$. For a finite set $F \subset K$, define*

$$P_F x = \sum_{k\in F} \langle x, \varphi_k\rangle \, \varphi_k.$$

(a) Show that $x - P_F x$ is orthogonal to every φ_j, $j \in F$.

(b) Deduce

$$\|x\|^2 = \sum_{k\in F} |\langle x, \varphi_k\rangle|^2 + \|x - P_F x\|^2 .$$

(c) Conclude Bessel's inequality

$$\boxed{\sum_{k\in K} |\langle x, \varphi_k\rangle|^2 \le \|x\|^2 .}$$

Solution

For $j \in F$,

$$\begin{aligned}\langle x - P_F x, \varphi_j\rangle &= \langle x, \varphi_j\rangle - \sum_{k\in F} \langle x, \varphi_k\rangle \langle \varphi_k, \varphi_j\rangle \\ &= \langle x, \varphi_j\rangle - \langle x, \varphi_j\rangle = 0.\end{aligned}$$

Therefore

$$x = P_F x + (x - P_F x)$$

is an orthogonal decomposition. By Pythagoras,

$$\|x\|^2 = \|P_F x\|^2 + \|x - P_F x\|^2 .$$

Since the φ_k are orthonormal,

$$\|P_F x\|^2 = \sum_{k\in F} |\langle x, \varphi_k\rangle|^2.$$

Hence

$$\boxed{\|x\|^2 = \sum_{k\in F} |\langle x, \varphi_k\rangle|^2 + \|x - P_F x\|^2 .}$$

Dropping the nonnegative residual gives

$$\sum_{k\in F} |\langle x, \varphi_k\rangle|^2 \le \|x\|^2 .$$

Taking larger and larger finite subsets F gives Bessel's inequality.

Conceptual message

Bessel is simply "captured energy $\le$ total energy". The missing energy is exactly the squared norm of the orthogonal residual.

3. Parseval equality and completeness

Exercise 3 (Parseval)**.** *Let $\{\varphi_k\}_{k\in K}$ be an orthonormal set in a Hilbert space H.*

(a) Show that if $\{\varphi_k\}$ is complete, then for every $x \in H$,

$$\boxed{\|x\|^2 = \sum_{k\in K} |\langle x, \varphi_k\rangle|^2.}$$

(b) Show conversely that if this equality holds for every $x \in H$, then the orthonormal set is complete.

(c) In $\mathbb{R}^3$, consider

$$\varphi_1 = \frac{1}{\sqrt{2}}(1,1,0)^T, \qquad \varphi_2 = \frac{1}{\sqrt{2}}(1,-1,0)^T,$$

and $x = (1,2,3)^T$. Verify Bessel's inequality explicitly, identify the missing energy, and then add one vector to obtain Parseval equality.

Solution

Let

$$S = \overline{\operatorname{span}\{\varphi_k : k \in K\}}.$$

The orthogonal expansion gives the projection $P_S x$. From the identity behind Bessel,

$$\|x\|^2 = \sum_k |\langle x, \varphi_k \rangle|^2 + \|x - P_S x\|^2 .$$

If the set is complete, then $S = H$, so $P_S x = x$ and the residual vanishes. Thus

$$\boxed{\|x\|^2 = \sum_k |\langle x, \varphi_k \rangle|^2.}$$

This is Parseval's equality.
Conversely, if Parseval holds for every x, then

$$\|x - P_S x\|^2 = 0$$

for every x, hence $P_S x = x$ and therefore $S = H$. So the orthonormal set is complete.
For the concrete example,

$$\langle x, \varphi_1 \rangle = \frac{3}{\sqrt{2}}, \qquad \langle x, \varphi_2 \rangle = -\frac{1}{\sqrt{2}}.$$

Thus

$$|\langle x, \varphi_1 \rangle|^2 + |\langle x, \varphi_2 \rangle|^2 = \frac{9}{2} + \frac{1}{2} = 5.$$

But

$$\|x\|^2 = 1 + 4 + 9 = 14,$$

so Bessel is strict:

$$5 < 14.$$

The residual is

$$x - P_S x = (0,0,3)^T,$$

whose energy is $9 = 14 - 5$. Adding

$$\varphi_3 = (0,0,1)^T$$

produces an orthonormal basis of $\mathbb{R}^3$, and now

$$5 + |\langle x, \varphi_3 \rangle|^2 = 5 + 9 = 14 = \|x\|^2 .$$

Conceptual message

For an orthonormal system:

$$\boxed{\text{Bessel inequality} + \text{completeness} = \text{Parseval equality.}}$$

4. Parseval and change of orthonormal basis

Exercise 4 (Inspired by FSP Exercise 2.35)**.** *Let $\{e_1, e_2\}$ be the standard basis of $\mathbb{R}^2$ and let*

$$\varphi_1 = \frac{1}{\sqrt{2}}(1,1)^T, \qquad \varphi_2 = \frac{1}{\sqrt{2}}(1,-1)^T.$$

For $x = e_1$:

(a) Compute the coefficient vector of x in both orthonormal bases.

(b) Compare the ℓ^1, ℓ^2 and ℓ^∞ norms of the two coefficient vectors.

(c) Explain why the equality of the ℓ^2 norms is not accidental.

Solution

In the standard basis,

$$x = 1\, e_1 + 0\, e_2, \qquad c = (1,0)^T.$$

In the rotated orthonormal basis,

$$\alpha_1 = \langle x, \varphi_1 \rangle = \frac{1}{\sqrt{2}}, \qquad \alpha_2 = \langle x, \varphi_2 \rangle = \frac{1}{\sqrt{2}},$$

so

$$\alpha = \frac{1}{\sqrt{2}}(1,1)^T.$$

Therefore

$$\|c\|_1 = 1, \qquad \|\alpha\|_1 = \sqrt{2},$$

while

$$\|c\|_\infty = 1, \qquad \|\alpha\|_\infty = \frac{1}{\sqrt{2}}.$$

But

$$\|c\|_2 = 1 = \|\alpha\|_2 .$$

The reason is Parseval: for any orthonormal basis $\{\varphi_k\}$,

$$\boxed{\|x\|^2 = \sum_k |\langle x, \varphi_k \rangle|^2.}$$

Thus the Euclidean norm of the coefficient vector is invariant under an orthonormal change of basis.

5. Riesz bases and stability

Exercise 5 (FSP Exercise 2.33, diagonal version)**.** *Let $\{e_k\}_{k\in\mathbb{Z}}$ be the standard orthonormal basis of $\ell^2(\mathbb{Z})$ and let*

$$\varphi_k = a_k e_k, \qquad a_k \neq 0.$$

(a) Show that $\{\varphi_k\}$ is a Riesz basis if and only if

$$0 < m := \inf_k |a_k| \leq \sup_k |a_k| =: M < \infty.$$

(b) Determine Riesz bounds in terms of m and M.

(c) Apply the criterion to the standard basis $a_k = 1$.

(d) Apply it to

$$\varphi_k = 2^k e_k, \qquad k \in \mathbb{Z},$$

and show that this is a basis but not a Riesz basis.

Solution

For a finitely supported sequence $c = (c_k)$,

$$\sum_k c_k \varphi_k = \sum_k c_k a_k e_k.$$

Since the e_k are orthonormal,

$$\left\| \sum_k c_k \varphi_k \right\|^2 = \sum_k |c_k|^2 |a_k|^2.$$

Hence, if

$$m \le |a_k| \le M,$$

then

$$\boxed{m^2 \sum_k |c_k|^2 \le \left\| \sum_k c_k \varphi_k \right\|^2 \le M^2 \sum_k |c_k|^2.}$$

Thus $A = m^2$ and $B = M^2$ are Riesz bounds.
Conversely, assume Riesz bounds A, B hold. Test the inequality with $c = e_n$. Then

$$A \le \|\varphi_n\|^2 = |a_n|^2 \le B$$

for every n. Therefore

$$\sqrt{A} \le |a_n| \le \sqrt{B},$$

which proves the criterion.
For $a_k = 1$,

$$m = M = 1,$$

so the standard basis is a Riesz basis with optimal bounds

$$\boxed{A = B = 1.}$$

For $a_k = 2^k$, every a_k is nonzero, and every $x = (x_k) \in \ell^2(\mathbb{Z})$ has the unique coordinate expansion

$$x = \sum_k x_k e_k = \sum_k (2^{-k} x_k) \varphi_k,$$

with the same partial sums in the ambient space. Hence $\{2^k e_k\}$ is a basis. However,

$$\inf_{k \in \mathbb{Z}} 2^k = 0, \qquad \sup_{k \in \mathbb{Z}} 2^k = \infty.$$

So no positive lower Riesz bound and no finite upper Riesz bound exist:

$$\boxed{\{2^k e_k\}_{k \in \mathbb{Z}} \text{ is a basis but not a Riesz basis.}}$$

Conceptual message

An orthonormal basis is perfectly conditioned: $A = B = 1$. A Riesz basis may distort energy, but only by fixed factors. An arbitrary basis can distort different coordinates by arbitrarily large or arbitrarily small amounts.

Common pitfall

"Basis" and "Riesz basis" are not synonyms in infinite dimension. Multiplying each basis vector by a nonzero scalar preserves the algebraic/coordinate representation, but if the scalings approach 0 or ∞, uniform stability is lost.

6. Backup: finite-dimensional Riesz bounds from a Gram matrix

Additional exercise

Consider in $\mathbb{R}^2$

$$\varphi_1 = (1,0)^T, \qquad \varphi_2 = (1,1)^T.$$

This is a basis, but it is not orthonormal. Find the best constants A, B such that

$$A(|c_1|^2 + |c_2|^2) \le \|c_1\varphi_1 + c_2\varphi_2\|_2^2 \le B(|c_1|^2 + |c_2|^2).$$

Solution. The Gram matrix is

$$G = \begin{pmatrix} \langle\varphi_1,\varphi_1\rangle & \langle\varphi_2,\varphi_1\rangle \\ \langle\varphi_1,\varphi_2\rangle & \langle\varphi_2,\varphi_2\rangle \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}.$$

For $c = (c_1, c_2)^T$,

$$\|c_1\varphi_1 + c_2\varphi_2\|^2 = c^T G c.$$

Therefore the optimal bounds are the smallest and largest eigenvalues of G:

$$\boxed{A = \frac{3-\sqrt{5}}{2}, \qquad B = \frac{3+\sqrt{5}}{2}.}$$

This is the finite-dimensional analogue of Riesz stability.

7. Backup: even and odd subspaces

Additional exercise

Inspired by FSP Exercise 2.36. In $L^2([-\pi,\pi])$, define

$$x_{\text{even}}(t) = \frac{x(t)+x(-t)}{2}, \qquad x_{\text{odd}}(t) = \frac{x(t)-x(-t)}{2}.$$

Show that

$$x = x_{\text{even}} + x_{\text{odd}}, \qquad x_{\text{even}} \perp x_{\text{odd}},$$

and identify orthonormal bases of the even and odd subspaces from the standard trigonometric basis.

Solution. First,

$$x_{\text{even}}(t) + x_{\text{odd}}(t) = \frac{x(t) + x(-t)}{2} + \frac{x(t) - x(-t)}{2} = x(t).$$

Moreover,

$$x_{\text{even}}(-t) = x_{\text{even}}(t), \qquad x_{\text{odd}}(-t) = -x_{\text{odd}}(t),$$

so the two terms belong respectively to the even and odd subspaces.
If f is even and g is odd, then $f(t)g(t)^*$ is odd. Hence

$$\langle f, g \rangle = \int_{-\pi}^{\pi} f(t)g(t)^* \, dt = 0,$$

which proves that the two subspaces are orthogonal. Therefore

$$\boxed{L^2([-\pi, \pi]) = S_{\text{even}} \oplus S_{\text{odd}}.}$$

Starting from the standard trigonometric orthonormal basis of $L^2([-\pi, \pi])$, the even functions are generated by the constant and cosine terms, while the odd functions are generated by the sine terms. Thus an orthonormal basis for the even subspace is

$$\boxed{\left\{ \frac{1}{\sqrt{2\pi}} \right\} \cup \left\{ \frac{1}{\sqrt{\pi}} \cos(kt) : k \geq 1 \right\},}$$

and an orthonormal basis for the odd subspace is

$$\boxed{\left\{ \frac{1}{\sqrt{\pi}} \sin(kt) : k \geq 1 \right\}.}$$

Completeness follows because together these two families form the complete trigonometric orthonormal basis of $L^2([-\pi, \pi])$.