Advanced Methods for Information Representation
Laboratory Session 1

Adjoints, Projections, Parseval, and Riesz Stability

University of Brescia – Department of Information Engineering

Learning goals

At the end of the laboratory you should be able to:

- recognize an adjoint numerically and understand the role of normalization;
- identify an orthogonal projector from the identities $P^2 = P$ and $P^* = P$;
- distinguish an orthogonal projection from an oblique projection onto the same subspace;
- verify Bessel's inequality and Parseval's identity numerically for an orthonormal basis;
- interpret Riesz bounds as quantitative stability bounds for coordinates.

The emphasis is on the mathematical decisions. Most plotting and bookkeeping code is already provided in the notebook.

Notebook workflow

Work through the notebook in order. Cells marked `TODO` require only a few lines of code. Before running a check, write down the mathematical identity that you expect to see numerically. Numerical equalities should be interpreted up to floating-point precision.

Exercise 1 – Local averaging, adjoint, and orthogonal projection

Consider a sampled signal $x \in \mathbb{R}^N$. Split its entries into M non-overlapping blocks of equal length

$$L = \frac{N}{M}.$$

Define the block-averaging operator $A_{\text{av}} : \mathbb{R}^N \to \mathbb{R}^M$ by

$$(A_{\text{av}}x)_k = \frac{1}{L} \sum_{n \in I_k} x_n,$$

where I_k is the kth block. Introduce the normalized operator

$$Q = \sqrt{L}\, A_{\text{av}}.$$

(a) Build A_{av} using the supplied helper code and form Q. Verify numerically that

$$QQ^* = I_M.$$

Explain why the normalization $\sqrt{L}$ is the correct one for the Euclidean inner product.

(b) Define the operator

$$P = Q^*Q = LA_{\text{av}}^* A_{\text{av}}.$$

Without explicitly forming a large $N \times N$ matrix, apply P to x and verify numerically

$$P(Px) = Px.$$

What does Px look like on each block I_k?

(c) Verify the orthogonality condition

$$\langle x - Px, Px \rangle \approx 0$$

and the Pythagorean identity

$$\|x\|_2^2 \approx \|Px\|_2^2 + \|x - Px\|_2^2 .$$

Interpret Px as the best approximation of x among signals that are constant on every block.

(d) Compare $A_{\mathrm{av}}x$ and Px in a plot. Relate the expression

$$Px = LA_{\mathrm{av}}^* A_{\mathrm{av}} x$$

to the sampling/reconstruction procedure discussed in class.

Conceptual checkpoint

The important object is not merely $A_{\mathrm{av}}^* A_{\mathrm{av}}$: the normalization matters. With $Q = \sqrt{L} A_{\mathrm{av}}$, the rows of Q are orthonormal, hence $QQ^* = I$ and Q^*Q is an orthogonal projector.

Exercise 2 – Orthogonal versus oblique projection

Let

$$B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad S = \mathrm{Ran}(B) \subset \mathbb{R}^3.$$

(a) Compute the orthogonal projector onto S using

$$P_\perp = B(B^T B)^{-1} B^T.$$

Check numerically that

$$P_\perp^2 = P_\perp, \qquad P_\perp^T = P_\perp.$$

(b) Consider

$$C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Verify that $CB = I_2$ and define

$$P_{\mathrm{obl}} = BC.$$

Show numerically that P_{obl} is a projection but is not self-adjoint.

(c) Verify that both projectors have the same range by checking that

$$P_\perp B = B, \qquad P_{\mathrm{obl}} B = B.$$

What differs between the two projections?

(d) For

$$x = (6, 6, 8)^T,$$

compute $P_\perp x$ and $P_{\mathrm{obl}} x$. Compare the errors

$$e_\perp = x - P_\perp x, \qquad e_{\mathrm{obl}} = x - P_{\mathrm{obl}} x.$$

Check that

$$B^T e_\perp = 0$$

while, in general, $B^T e_{\mathrm{obl}} \neq 0$. Which projection gives the smaller approximation error?

Conceptual checkpoint

Idempotence $P^2 = P$ says that P is a projection. Self-adjointness $P^* = P$ forces the projection direction to be orthogonal to the target subspace. Orthogonal projection is therefore the unique best approximation in Euclidean norm.

Exercise 3 – Orthonormal coordinates: Bessel and Parseval

Consider the normalized Hadamard matrix

$$U = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} = \begin{bmatrix} u_1 & u_2 & u_3 & u_4 \end{bmatrix}.$$

Its columns form an orthonormal basis of $\mathbb{R}^4$.

Let

$$x = (2, -1, 3, 4)^T.$$

(a) Verify numerically that

$$U^T U = I_4.$$

Compute the analysis coefficients

$$\alpha = U^T x,$$

and reconstruct x from $U\alpha$.

(b) Verify Parseval's identity

$$\boxed{\|x\|_2^2 = \sum_{k=1}^{4} |\alpha_k|^2.}$$

Interpret this equality as conservation of energy under an orthonormal change of coordinates.

(c) For $m = 1, 2, 3$, define the partial reconstruction

$$x_m = \sum_{k=1}^{m} \alpha_k u_k.$$

Verify Bessel's inequality

$$\sum_{k=1}^{m} |\alpha_k|^2 \leq \|x\|_2^2$$

and the stronger identity

$$\|x - x_m\|_2^2 = \|x\|_2^2 - \sum_{k=1}^{m} |\alpha_k|^2.$$

What happens when $m = 4$?

(d) Compare this situation with the non-orthogonal basis

$$\Phi_{1/2} = \begin{pmatrix} 1 & 1 \\ 0 & 1/2 \end{pmatrix}.$$

For a vector $z \in \mathbb{R}^2$, compute its coefficient vector $c = \Phi_{1/2}^{-1} z$. Check that, in general,

$$\|z\|_2^2 \neq \|c\|_2^2.$$

This motivates replacing Parseval equality by two-sided Riesz bounds.

Conceptual checkpoint

For an orthonormal basis, analysis preserves energy exactly: Parseval gives $\|x\|_2 = \|\alpha\|_2$. For an incomplete orthonormal family one only has Bessel's inequality. For a non-orthogonal basis exact energy preservation is generally lost; Riesz bounds quantify how far from an isometry the coordinate map can be.

Exercise 4 – Riesz bases as stable coordinate systems

For a finite family collected as the columns of a square invertible matrix Φ, the Riesz inequalities are

$$A\,\|c\|_2^2 \le \|\Phi c\|_2^2 \le B\,\|c\|_2^2 \qquad \text{for every coefficient vector } c.$$

The optimal bounds are

$$A = \lambda_{\min}(\Phi^*\Phi), \qquad B = \lambda_{\max}(\Phi^*\Phi).$$

Notice that an orthonormal basis is the special case $A = B = 1$, i.e. Parseval.

4A. A nearly degenerate basis in $\mathbb{R}^2$

For $\varepsilon > 0$, let

$$\Phi_\varepsilon = \begin{pmatrix} 1 & 1 \\ 0 & \varepsilon \end{pmatrix} = \begin{bmatrix} \varphi_1 & \varphi_2 \end{bmatrix}.$$

(a) For $\varepsilon \in \{1, 0.5, 0.1, 0.02\}$, compute the optimal Riesz bounds $A_\varepsilon, B_\varepsilon$ and the ratio $B_\varepsilon/A_\varepsilon$.

(b) What happens as the two basis vectors become almost parallel?

(c) Choose a fixed coefficient vector c, form $x = \Phi_\varepsilon c$, perturb x by a small vector, and recover the coefficients by solving a linear system. Compare the relative perturbation in x with the relative perturbation in the recovered coefficients.

4B. Finite truncations of $\varphi_k = 2^k e_k$

For $K \ge 1$, define a finite-dimensional model indexed by $k = -K, \dots, K$:

$$\Phi_K = \operatorname{diag}(2^{-K}, 2^{-K+1}, \dots, 2^K).$$

(a) Compute the optimal Riesz bounds A_K, B_K for several values of K.

(b) Plot A_K, B_K and B_K/A_K on a logarithmic scale.

(c) Use the numerical experiment to explain why the infinite family $\{2^k e_k\}_{k\in\mathbb{Z}}$ is a basis of $\ell^2(\mathbb{Z})$ but not a Riesz basis.

Conceptual checkpoint

In finite dimension every basis has positive finite Riesz bounds. The relevant infinite-dimensional question is whether the bounds remain uniform as the dimension grows. For $\varphi_k = 2^k e_k$, the finite truncations become arbitrarily ill-conditioned, so no positive lower and finite upper bound can hold uniformly.