Advanced Methods for Information Representation
Vector Spaces I: Basics, Inner Products, Norms, and Distances

Course notes

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1 Why vector spaces in signal processing?

The basic objects of signal processing may look very different: a finite block of samples, a two-sided discrete-time sequence, a continuous-time waveform, an image, or a random variable. The vector-space viewpoint places these objects in a common mathematical framework. The essential operations are addition and scaling; once an inner product and a norm are added, familiar Euclidean ideas such as length, angle, orthogonality, approximation, and distance extend to signals.

These notes follow the order of the course slides and use the notation of Chapter 2 of *Foundations of Signal Processing* (FSP). In particular,

$$\mathbb{C}^N, \qquad \ell^p(\mathbb{Z}), \qquad L^p(\mathbb{R}), \qquad \langle x, y\rangle, \qquad \|x\|_p$$

are used throughout. The inner product is linear in its first argument and conjugate-linear in its second.

2 Groups and fields

2.1 Groups

Definition 2.1 (Group)**.** A *group* is a pair $(G, *)$, where G is a set and $*$ is a binary operation on G, such that:

(i) $(a * b) * c = a * (b * c)$ for all $a, b, c \in G$;

(ii) there exists an identity element $e \in G$ satisfying $a * e = e * a = a$;

(iii) every $a \in G$ has an inverse $a^{-1} \in G$ satisfying $a * a^{-1} = a^{-1} * a = e$.

If $a * b = b * a$ for all $a, b \in G$, the group is called *abelian* or *commutative*.

Example 2.2. $(\mathbb{Z}, +)$, $(\mathbb{R}, +)$ and $(\mathbb{C}, +)$ are abelian groups. The set of invertible $N \times N$ matrices with matrix multiplication is a group, but for $N \geq 2$ it is generally not abelian.

2.2 Fields

Definition 2.3 (Field)**.** A *field* $\mathbb{K}$ is a set equipped with addition and multiplication such that $(\mathbb{K}, +)$ is an abelian group, $(\mathbb{K} \setminus \{0\}, \cdot)$ is an abelian group, and multiplication distributes over addition.

The two fields used in this course are $\mathbb{R}$ and $\mathbb{C}$. Elements of the field are called *scalars*.

3 Vector spaces

3.1 Definition

Definition 3.1 (Vector space)**.** Let $\mathbb{K}$ be a field. A set V is a *vector space over* $\mathbb{K}$ if it is equipped with vector addition and scalar multiplication satisfying, for all $x, y, z \in V$ and $\alpha, \beta \in \mathbb{K}$,

$$\begin{aligned} x + y &= y + x, & (x + y) + z &= x + (y + z), \\ x + 0 &= x, & x + (-x) &= 0, \\ \alpha(x + y) &= \alpha x + \alpha y, & (\alpha + \beta)x &= \alpha x + \beta x, \\ (\alpha\beta)x &= \alpha(\beta x), & 1x &= x. \end{aligned}$$

Two immediate identities are useful.

Proposition 3.2. *For every $x \in V$ and $\alpha \in \mathbb{K}$,*

$$0x = 0, \qquad \alpha 0 = 0,$$

where the first 0 in $0x$ is the scalar zero and the second is the zero vector.

Proof. Since $(0 + 0)x = 0x + 0x$ and $(0 + 0)x = 0x$, adding the additive inverse of $0x$ gives $0x = 0$. Similarly, $\alpha(0 + 0) = \alpha 0 + \alpha 0$ implies $\alpha 0 = 0$. □

3.2 Basic examples

Example 3.3 (Finite-dimensional vectors)**.**

$$\mathbb{C}^N = \{x = (x_0, x_1, \ldots, x_{N-1})^T : x_n \in \mathbb{C}\}$$

is a vector space over $\mathbb{C}$. Addition and scalar multiplication are componentwise.

Example 3.4 (Two-sided sequences)**.**

$$\mathbb{C}^{\mathbb{Z}} = \{x = (\ldots, x_{-1}, x_0, x_1, \ldots) : x_n \in \mathbb{C}\}$$

is a vector space over $\mathbb{C}$. In signal-processing notation, x_n is the nth sample.

Example 3.5 (Functions)**.** The set

$$\mathbb{C}^{\mathbb{R}} = \{x : \mathbb{R} \to \mathbb{C}\}$$

is a vector space under pointwise operations

$$(x + y)(t) = x(t) + y(t), \qquad (\alpha x)(t) = \alpha x(t).$$

Likewise, one may consider functions defined on $[a, b]$, on $\mathbb{R}_+$, or on other domains.

Example 3.6 (Polynomials)**.** The set

$$\mathcal{P}_{N-1} = \left\{ x(t) = \sum_{n=0}^{N-1} \alpha_n t^n : \alpha_n \in \mathbb{C} \right\}$$

of polynomials of degree at most $N - 1$ is a vector space over $\mathbb{C}$.

4 Subspaces and affine subspaces

4.1 Linear subspaces

Definition 4.1 (Subspace)**.** A nonempty subset S of a vector space V is a *subspace* if it is closed under linear combinations:

$$x, y \in S,\ \alpha, \beta \in \mathbb{K} \quad \Longrightarrow \quad \alpha x + \beta y \in S.$$

Example 4.2. Fix $x_0 \in V$. Then

$$S = \{\alpha x_0 : \alpha \in \mathbb{K}\}$$

is a one-dimensional subspace whenever $x_0 \neq 0$.

Example 4.3 (Odd functions)**.** The set

$$S_{\text{odd}} = \{x \in \mathbb{C}^{\mathbb{R}} : x(t) = -x(-t)\}$$

is a subspace. Indeed, if x, y are odd, then

$$(\alpha x + \beta y)(-t) = \alpha x(-t) + \beta y(-t) = -(\alpha x(t) + \beta y(t)).$$

4.2 Affine subspaces

Definition 4.4 (Affine subspace)**.** Let S be a linear subspace of V and let $v_0 \in V$. The translated set

$$T = v_0 + S = \{v_0 + s : s \in S\}$$

is an *affine subspace*.

Proposition 4.5. *An affine subspace T is a linear subspace if and only if $0 \in T$.*

Proof. If T is a subspace then it contains 0. Conversely, if $T = v_0 + S$ and $0 \in T$, then $v_0 = -s_0$ for some $s_0 \in S$. Thus

$$T = v_0 + S = -s_0 + S = S.$$

□

Example 4.6. In $\mathbb{R}^2$, the line $\{(t, 1) : t \in \mathbb{R}\}$ is affine but not linear. The line $\{(t, 0) : t \in \mathbb{R}\}$ is linear.

5 Span, linear independence, and dimension

5.1 Span

Definition 5.1 (Span)**.** For $S \subseteq V$, the span of S is the set of all finite linear combinations of vectors in S:

$$\operatorname{span}(S) = \left\{ \sum_{k=1}^{m} \alpha_k \phi_k : m < \infty,\ \alpha_k \in \mathbb{K},\ \phi_k \in S \right\}.$$

Proposition 5.2. $\operatorname{span}(S)$ *is the smallest subspace of V that contains S.*

Proof. Finite linear combinations are stable under addition and scalar multiplication, so $\operatorname{span}(S)$ is a subspace. It contains S because each $\phi \in S$ equals 1ϕ. Any subspace containing S must contain every finite linear combination of elements of S. □

5.2 Linear independence

Definition 5.3 (Linear independence)**.** A family $\{\phi_k\}_{k\in K}$ is linearly independent if every finite relation

$$\sum_{k\in F} \alpha_k \phi_k = 0, \qquad F \subset K \text{ finite},$$

has only the trivial solution $\alpha_k = 0$ for all $k \in F$.

5.3 Dimension

Definition 5.4 (Basis and dimension)**.** A basis of a finite-dimensional vector space is a linearly independent spanning set. The number of vectors in a basis is the dimension of the space.

Example 5.5. The standard vectors $e_0, \dots, e_{N-1}$ form a basis of $\mathbb{C}^N$, hence $\dim \mathbb{C}^N = N$. The monomials $1, t, \dots, t^{N-1}$ form a basis of $\mathcal{P}_{N-1}$, hence $\dim \mathcal{P}_{N-1} = N$.

Spaces of arbitrary two-sided sequences and general functions are infinite-dimensional.

6 Inner products and orthogonality

6.1 Inner products

Definition 6.1 (Inner product)**.** Let V be a vector space over $\mathbb{C}$. An inner product is a map

$$\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C}$$

such that, for all $x, y, z \in V$ and $\alpha \in \mathbb{C}$,

(i) $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$;

(ii) $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$;

(iii) $\langle x, y \rangle = \langle y, x \rangle^*$;

(iv) $\langle x, x \rangle \geq 0$, with equality if and only if $x = 0$.

Hermitian symmetry and linearity in the first argument imply conjugate-linearity in the second:

$$\langle x, \alpha y \rangle = \alpha^* \langle x, y \rangle .$$

For real vector spaces the conjugation disappears.

Example 6.2 (Weighted inner product)**.** On $\mathbb{C}^2$,

$$\langle x, y \rangle = x_0 y_0^* + 5 x_1 y_1^*$$

is an inner product. The positive weight changes angles and lengths but not the vector-space structure.

6.2 Standard inner products

The spaces most frequently used in FSP are

$$\mathbb{C}^N : \qquad \langle x, y \rangle = \sum_{n=0}^{N-1} x_n y_n^*, \tag{1}$$

$$\ell^2(\mathbb{Z}) : \qquad \langle x, y \rangle = \sum_{n\in\mathbb{Z}} x_n y_n^*, \tag{2}$$

$$L^2(\mathbb{R}) : \qquad \langle x, y \rangle = \int_{-\infty}^{\infty} x(t) y(t)^* \, dt. \tag{3}$$

The notation ℓ^2 is used for sequences and L^2 for functions.

Definition 6.3 (Orthogonality)**.** Vectors x, y are orthogonal, written $x \perp y$, if $\langle x, y\rangle = 0$. A family is orthogonal if distinct members are mutually orthogonal, and orthonormal if additionally every member has unit norm:

$$\langle \phi_k, \phi_j\rangle = \delta_{kj}.$$

Definition 6.4 (Orthogonal complement)**.** For a set S in an inner product space,

$$S^{\perp} = \{x : \langle x, s\rangle = 0 \text{ for every } s \in S\}.$$

Proposition 6.5. *$S^{\perp}$ is a subspace.*

Proof. If $x, y \in S^{\perp}$ and $\alpha, \beta \in \mathbb{K}$, then for every $s \in S$,

$$\langle \alpha x + \beta y, s\rangle = \alpha \langle x, s\rangle + \beta \langle y, s\rangle = 0.$$

□

Proposition 6.6. *Every orthonormal family is linearly independent.*

Proof. Suppose $\sum_{k=1}^{m} \alpha_k \phi_k = 0$. Taking the inner product with ϕ_j gives

$$0 = \sum_{k=1}^{m} \alpha_k \langle \phi_k, \phi_j\rangle = \alpha_j.$$

Since j was arbitrary, all coefficients vanish. □

6.3 A trigonometric example

On $[-1/2, 1/2]$ consider

$$\phi_0(t) = 1, \qquad \phi_k(t) = \sqrt{2}\cos(2\pi k t), \quad k \geq 1.$$

With the inner product

$$\langle x, y\rangle = \int_{-1/2}^{1/2} x(t) y(t)^{*}\, dt,$$

the family $\{\phi_k\}_{k\geq 0}$ is orthonormal. For $k, j \geq 1$ this follows from

$$2\cos(2\pi k t)\cos(2\pi j t) = \cos(2\pi(k-j)t) + \cos(2\pi(k+j)t).$$

The same family is orthogonal to every odd function on the symmetric interval because an even function times an odd function is odd, and an odd function integrates to zero over a symmetric interval.

Definition 6.7 (Inner product space)**.** A vector space equipped with an inner product is called an *inner product space*.

7 Norms

7.1 Definition

Definition 7.1 (Norm)**.** A norm on V is a map $\|\cdot\| : V \to [0, \infty)$ satisfying

(i) $\|x\| = 0$ if and only if $x = 0$;

(ii) $\|\alpha x\| = |\alpha|\,\|x\|$;

(iii) $\|x + y\| \leq \|x\| + \|y\|$.

Every inner product induces a norm by

$$\|x\| = \sqrt{\langle x, x\rangle}.$$

Not every norm is induced by an inner product.

Example 7.2. For $x = (x_0, x_1)^T \in \mathbb{C}^2$, all of the following are norms:

$$(|x_0|^2 + 5|x_1|^2)^{1/2}, \qquad |x_0| + |x_1|, \qquad \max(|x_0|, |x_1|).$$

The quantity $|x_0|$ is not a norm on $\mathbb{C}^2$ because it vanishes for nonzero vectors such as $(0, 1)^T$.

7.2 Standard 2-norms

The inner products in (1)–(3) induce

$$\|x\|_2 = \left(\sum_{n=0}^{N-1} |x_n|^2\right)^{1/2} \qquad \text{on } \mathbb{C}^N,$$

$$\|x\|_2 = \left(\sum_{n\in\mathbb{Z}} |x_n|^2\right)^{1/2} \qquad \text{on } \ell^2(\mathbb{Z}),$$

$$\|x\|_2 = \left(\int_{\mathbb{R}} |x(t)|^2\, dt\right)^{1/2} \qquad \text{on } L^2(\mathbb{R}).$$

In signal processing, $\|x\|_2^2$ is commonly interpreted as signal energy.

8 Geometry induced by an inner product

8.1 Pythagorean theorem

Theorem 8.1 (Pythagorean theorem)**.** *If $x \perp y$, then*

$$\|x + y\|^2 = \|x\|^2 + \|y\|^2 .$$

More generally, for a finite mutually orthogonal family $\{x_k\}$,

$$\left\|\sum_k x_k\right\|^2 = \sum_k \|x_k\|^2 .$$

Proof. Expand the square using the inner product:

$$\begin{aligned}\|x + y\|^2 &= \langle x + y, x + y\rangle \\ &= \|x\|^2 + \|y\|^2 + \langle x, y\rangle + \langle y, x\rangle .\end{aligned}$$

The last two terms vanish when $x \perp y$. □

8.2 Parallelogram law

Theorem 8.2 (Parallelogram law)**.** *For all x, y in an inner product space,*

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2 .$$

Proof. Expand both squared norms using the inner product. The cross terms have opposite signs and cancel when the two expressions are added. □

Remark 8.3. The parallelogram law is also a characterization: a norm comes from an inner product if and only if it satisfies the parallelogram identity. The converse is known as the Jordan–von Neumann theorem.

8.3 Cauchy–Schwarz inequality

Theorem 8.4 (Cauchy–Schwarz)**.** *For all x, y in an inner product space,*

$$|\langle x, y\rangle| \le \|x\| \, \|y\| \, .$$

Equality holds if and only if x and y are linearly dependent.

Proof. If $x = 0$ the result is immediate. Assume $x \neq 0$. For any $\alpha \in \mathbb{C}$,

$$0 \le \|y - \alpha x\|^2 \, .$$

Choose

$$\alpha = \frac{\langle y, x\rangle}{\|x\|^2}.$$

Then direct expansion gives

$$0 \le \|y\|^2 - \frac{|\, \langle x, y\rangle \,|^2}{\|x\|^2},$$

so $|\, \langle x, y\rangle \,|^2 \le \|x\|^2 \, \|y\|^2$. □

In a real inner product space, Cauchy–Schwarz guarantees that

$$-1 \le \frac{\langle x, y\rangle}{\|x\| \, \|y\|} \le 1,$$

so one may define the angle θ by

$$\cos\theta = \frac{\langle x, y\rangle}{\|x\| \, \|y\|}.$$

9 Normed spaces and metrics

Definition 9.1 (Normed vector space)**.** A vector space equipped with a norm is a *normed vector space*.

In infinite-dimensional spaces we usually restrict the ambient set to the vectors for which the chosen norm is finite. This is the origin of spaces such as ℓ^p and L^p.

Definition 9.2 (Metric)**.** A metric on a set V is a function $d : V \times V \to [0, \infty)$ satisfying

$$\begin{aligned} &d(x, y) = 0 \iff x = y, \\ &d(x, y) = d(y, x), \\ &d(x, y) \le d(x, z) + d(z, y). \end{aligned}$$

Every norm induces a metric by

$$d(x, y) = \|x - y\| \, .$$

Example 9.3 (A metric not induced by a norm)**.** On $\mathbb{R}$ define

$$d(x, y) = |\arctan x - \arctan y|.$$

This is a metric because it is the ordinary distance after applying the injective map $x \mapsto \arctan x$. It cannot be induced by a norm on $\mathbb{R}$ because every norm-induced distance is translation invariant:

$$d(x + a, y + a) = d(x, y),$$

whereas the arctangent metric is not.

10 Standard signal spaces

The central inner product spaces are

$$\mathbb{C}^N, \qquad \ell^2(\mathbb{Z}), \qquad L^2(\mathbb{R}), \qquad L^2([a,b]).$$

Other useful vector spaces include $C([a,b])$, spaces $C^q([a,b])$ of functions with continuous derivatives through order q, polynomial spaces, and spaces of random variables.

10.1 Random variables as vectors

Let X, Y be complex random variables with finite second moments. Define

$$\langle X, Y \rangle = \mathbb{E}[XY^*].$$

After identifying random variables that are equal almost surely, this is an inner product and

$$\|X\|_2 = (\mathbb{E}|X|^2)^{1/2}.$$

If $\mathbb{E}[X] = 0$, this norm is the standard deviation of X.

11 p-norms in finite dimension

For $x \in \mathbb{C}^N$ and $1 \le p < \infty$ define

$$\|x\|_p = \left(\sum_{n=0}^{N-1} |x_n|^p \right)^{1/p},$$

and

$$\|x\|_\infty = \max_{0 \le n < N} |x_n|.$$

The cases $p = 1$ and $p = 2$ are known as the Manhattan (or taxicab) and Euclidean norms, respectively.

Remark 11.1 ($0 < p < 1$)**.** The formula

$$\left(\sum_n |x_n|^p \right)^{1/p}$$

for $0 < p < 1$ does not satisfy the triangle inequality and hence is not a norm. In sparse signal processing, the notation $\|x\|_0$ is often used for the number of nonzero components of x; this is also not a norm.

Example 11.2 (Failure of the triangle inequality for $p = 1/2$)**.** Let $x = (1,0)^T$, $y = (0,1)^T$. Then

$$\|x\|_{1/2} = \|y\|_{1/2} = 1, \qquad \|x + y\|_{1/2} = (1+1)^2 = 4 > 2.$$

Proposition 11.3 (Equivalence of norms in finite dimension)**.** *On a finite-dimensional vector space, any two norms $\|\cdot\|_a$ and $\|\cdot\|_b$ are equivalent: there exist constants $c, C > 0$ such that*

$$c\,\|x\|_a \le \|x\|_b \le C\,\|x\|_a \qquad \textit{for all } x.$$

Thus all norms on $\mathbb{C}^N$ define the same notion of convergence, although their unit balls have different geometry. In $\mathbb{R}^2$, the unit ball is a diamond for $p = 1$, a circle for $p = 2$, and a square for $p = \infty$. Only the Euclidean 2-norm is invariant under arbitrary rotations of the coordinate system.

12 Sequence spaces $\ell^p(\mathbb{Z})$

For $1 \le p < \infty$,

Definition 12.1 ($\ell^p(\mathbb{Z})$)**.**

$$\ell^p(\mathbb{Z}) = \left\{ x : \mathbb{Z} \to \mathbb{C} : \sum_{n\in\mathbb{Z}} |x_n|^p < \infty \right\}$$

with norm

$$\|x\|_p = \left(\sum_{n\in\mathbb{Z}} |x_n|^p \right)^{1/p}.$$

For $p = \infty$,

$$\ell^\infty(\mathbb{Z}) = \{x : \sup_{n\in\mathbb{Z}} |x_n| < \infty\}, \qquad \|x\|_\infty = \sup_{n\in\mathbb{Z}} |x_n|.$$

Proposition 12.2 (Inclusion of sequence spaces)**.** *If $1 \le p < q \le \infty$, then*

$$\ell^p(\mathbb{Z}) \subseteq \ell^q(\mathbb{Z}).$$

Proof. Suppose first that $q < \infty$ and $x \in \ell^p$. Since $\sum |x_n|^p < \infty$, necessarily $x_n \to 0$. Hence $|x_n| \le 1$ except for finitely many indices. On those remaining indices, $|x_n|^q \le |x_n|^p$. Therefore $\sum |x_n|^q < \infty$. The case $q = \infty$ follows because every ℓ^p sequence is bounded. □

Example 12.3 (ℓ^2 but not ℓ^1)**.** Let

$$x_n = \begin{cases} 1/n, & n \ge 1, \\ 0, & n \le 0. \end{cases}$$

Then

$$\|x\|_2^2 = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}, \qquad \|x\|_2 = \frac{\pi}{\sqrt{6}},$$

while

$$\|x\|_1 = \sum_{n=1}^{\infty} \frac{1}{n} = \infty.$$

Thus $x \in \ell^2(\mathbb{Z})$ but $x \notin \ell^1(\mathbb{Z})$.

13 Function spaces $L^p(\mathbb{R})$

For $1 \le p < \infty$,

Definition 13.1 ($L^p(\mathbb{R})$)**.** $L^p(\mathbb{R})$ is the space of measurable complex-valued functions, identified up to equality almost everywhere, such that

$$\|x\|_p = \left(\int_{-\infty}^{\infty} |x(t)|^p \, dt \right)^{1/p} < \infty.$$

For $p = \infty$,

$$\|x\|_\infty = \operatorname*{ess\,sup}_{t\in\mathbb{R}} |x(t)|.$$

Remark 13.2 (No general inclusion on $\mathbb{R}$)**.** A key difference from sequence spaces is that, on the whole real line, $L^p(\mathbb{R})$ and $L^q(\mathbb{R})$ are generally not comparable when $p \ne q$. In particular, neither $L^1(\mathbb{R}) \subset L^2(\mathbb{R})$ nor $L^2(\mathbb{R}) \subset L^1(\mathbb{R})$ holds in general. This is explicitly emphasized in FSP when discussing Fourier-transform existence.

Example 13.3. The function

$$x(t) = \frac{1}{t}\,\mathbf{1}_{[1,\infty)}(t)$$

belongs to $L^2(\mathbb{R})$ but not to $L^1(\mathbb{R})$, since

$$\int_1^\infty \frac{dt}{t^2} < \infty, \qquad \int_1^\infty \frac{dt}{t} = \infty.$$

Conversely,

$$y(t) = t^{-1/2}\mathbf{1}_{(0,1)}(t)$$

belongs to $L^1(\mathbb{R})$ but not to $L^2(\mathbb{R})$.

Remark 13.4 (Finite intervals)**.** On a finite interval $[a, b]$, the situation changes: if $q > p$, then

$$L^q([a, b]) \subseteq L^p([a, b]).$$

This follows from Hölder's inequality. Thus the measure of the underlying domain matters.

14 Convergence in norm

Definition 14.1 (Norm convergence)**.** A sequence $\{x_k\}$ in a normed vector space V converges to $v \in V$ if

$$\lim_{k\to\infty} \|x_k - v\| = 0.$$

Equivalently, for every $\varepsilon > 0$ there exists K_ε such that

$$k > K_\varepsilon \quad \Longrightarrow \quad \|x_k - v\| < \varepsilon.$$

Convergence depends on the norm.

Example 14.2 (Convergence in L^p but not in L^∞)**.** Let

$$x_k(t) = \begin{cases} 1, & 0 \le t \le 1/k, \\ 0, & \text{otherwise.} \end{cases}$$

Then for $1 \le p < \infty$,

$$\|x_k\|_p^p = \int_0^{1/k} 1\,dt = \frac{1}{k}, \qquad \|x_k\|_p = k^{-1/p} \to 0.$$

Thus $x_k \to 0$ in every finite L^p norm. However,

$$\|x_k\|_\infty = 1$$

for every k, so there is no convergence to zero in L^∞.

15 Closed sets, closed subspaces, and closed span

Definition 15.1 (Closed set)**.** A subset S of a normed space is closed if every norm-convergent sequence $x_k \in S$ has its limit in S.

Definition 15.2 (Closure)**.** The closure $\overline{S}$ is the set of all limits of convergent sequences with terms in S.

Every finite-dimensional subspace of a normed vector space is closed. Infinite-dimensional spans need not be closed.

Definition 15.3 (Closed span)**.** For a family $\{\phi_k\}$, its closed span is

$$\overline{\operatorname{span}(\{\phi_k\})} = \overline{\operatorname{span}\{\phi_k\}}.$$

Equivalently, it is the set of norm-limits of sequences of finite linear combinations of the ϕ_k.

Example 15.4 (Finite-support sequences)**.** Let c_{00} be the set of all finitely supported sequences. It is a subspace of $\ell^2(\mathbb{Z})$ but is not closed. If $x \in \ell^2(\mathbb{Z})$, define the truncation

$$x_n^{(N)} = \begin{cases} x_n, & |n| \le N, \\ 0, & |n| > N. \end{cases}$$

Then $x^{(N)} \in c_{00}$ and

$$\left\| x - x^{(N)} \right\|_2^2 = \sum_{|n|>N} |x_n|^2 \to 0.$$

Therefore $\overline{c_{00}} = \ell^2(\mathbb{Z})$.

16 Cauchy sequences, completeness, Banach and Hilbert spaces

Definition 16.1 (Cauchy sequence)**.** A sequence $\{x_k\}$ in a normed space is Cauchy if for every $\varepsilon > 0$ there exists K_ε such that

$$k, m > K_\varepsilon \quad \Longrightarrow \quad \|x_k - x_m\| < \varepsilon.$$

Every convergent sequence is Cauchy. The converse need not hold unless the space is complete.

Definition 16.2 (Complete normed space)**.** A normed vector space is complete if every Cauchy sequence converges to an element of the space.

Definition 16.3 (Banach and Hilbert spaces)**.** A complete normed vector space is a *Banach space*. A complete inner product space is a *Hilbert space*.

Example 16.4 (An incomplete space)**.** $\mathbb{Q}$ with the ordinary absolute-value norm is not complete. A sequence of rational approximations to $\sqrt{2}$ is Cauchy in $\mathbb{Q}$ but does not converge to an element of $\mathbb{Q}$.

16.1 Standard complete spaces

The following facts are fundamental.

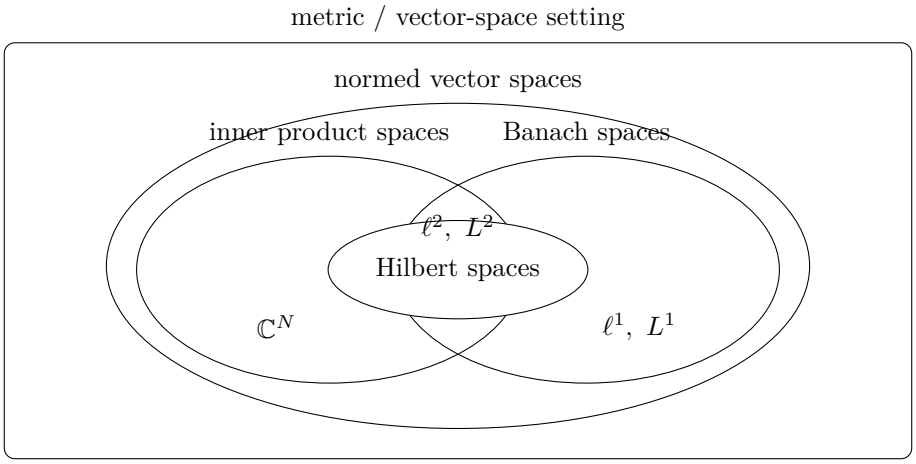
- Every finite-dimensional normed vector space is complete.
- $\ell^p(\mathbb{Z})$ is Banach for every $1 \le p \le \infty$.
- $L^p(\mathbb{R})$ is Banach for every $1 \le p \le \infty$.
- $\ell^2(\mathbb{Z})$ and $L^2(\mathbb{R})$ are Hilbert spaces because their norms are induced by inner products.
- $C([a,b])$ with the uniform norm $\|x\|_\infty = \max_{t\in[a,b]} |x(t)|$ is Banach.
- $C([a,b])$ equipped with a finite L^p norm is not complete: a Cauchy sequence of continuous functions may converge in L^p to a discontinuous function.

Remark 16.5. The statement "a space is complete" is incomplete unless the norm (or metric) is specified. Completeness is a property of a space together with its notion of distance.

17 How the spaces fit together

The main logical relations are:

- every inner product induces a norm;
- every norm induces a metric;
- Hilbert spaces are complete inner product spaces;
- Banach spaces are complete normed spaces;
- not every norm is induced by an inner product;
- not every normed space is complete.



The picture is schematic: a given set may carry different norms, and therefore its position in such a diagram depends on the chosen structure.

18 Separability

Definition 18.1 (Dense subset)**.** A subset D of a normed space V is dense if $\overline{D} = V$. Equivalently, every $x \in V$ can be approximated arbitrarily well by elements of D.

Definition 18.2 (Separable space)**.** A normed space is separable if it contains a countable dense subset.

Example 18.3. $\mathbb{R}$ is separable because $\mathbb{Q}$ is countable and dense in $\mathbb{R}$.

Theorem 18.4 (Separable Hilbert spaces)**.** *A Hilbert space is separable if and only if it possesses a finite or countably infinite orthonormal basis.*

This fact is central to signal representation. A separable Hilbert space admits a countable coordinate description: once an orthonormal basis is chosen, a vector is represented by a sequence of coefficients. Later in the course this observation becomes the bridge from abstract signal spaces to analysis/synthesis systems, Fourier expansions, filter banks, and wavelets.

19 Summary

The conceptual chain developed in this lecture is

$$\boxed{\text{vector space}} \longrightarrow \boxed{\text{inner product}} \longrightarrow \boxed{\text{norm}} \longrightarrow \boxed{\text{metric}}.$$

The analytic layer is added through convergence and completeness:

$$\boxed{\text{Cauchy sequences}} \longrightarrow \boxed{\text{completeness}} \longrightarrow \boxed{\text{Banach / Hilbert spaces}}.$$

For signal processing, the most important examples are $\mathbb{C}^N$, $\ell^2(\mathbb{Z})$ and $L^2(\mathbb{R})$. The last two are infinite-dimensional Hilbert spaces and provide the natural setting for energy signals, orthogonality, projections, and basis expansions.

Short exercises

Check your understanding 19.1. Show that the set of even functions in $L^2([-1,1])$ is a subspace and that it is orthogonal to the set of odd functions.

Check your understanding 19.2. Determine whether

$$S = \{(x_0, x_1, x_2)^T \in \mathbb{R}^3 : x_0 + x_1 + x_2 = 1\}$$

is a subspace or an affine subspace.

Check your understanding 19.3. For $x = (1, -1, 2)^T$, compute $\|x\|_1$, $\|x\|_2$, and $\|x\|_\infty$.

Check your understanding 19.4. Give an example of a function in $L^1(\mathbb{R}) \setminus L^2(\mathbb{R})$ and an example in $L^2(\mathbb{R}) \setminus L^1(\mathbb{R})$.

Check your understanding 19.5. Let e_k be the standard unit sequence in $\ell^2(\mathbb{Z})$. Show that $\{e_k\}_{k\in\mathbb{Z}}$ is orthonormal and that the set of its finite linear combinations is dense in $\ell^2(\mathbb{Z})$.

Reference

The main reference and notational basis for these notes is:

M. Vetterli, J. Kovačević, and V. K. Goyal, *Foundations of Signal Processing*, Cambridge University Press, 2014, Chapter 2, especially Sections 2.2 (Vector spaces) and 2.3 (Hilbert spaces). Free version: https://foundationsofsignalprocessing.org/FSP_v1.1_2014.pdf.