Esercitazione 1

Notazione

$$\mathrm{tri}(t) = \begin{cases} 1-|t| & |t|\le 1 \\ 0 & \text{altrimenti} \end{cases} \qquad \mathrm{rect}(t) = \begin{cases} 1 & |t| < \frac{1}{2} \\ 0 & |t| > \frac{1}{2} \end{cases}$$

$$\varepsilon(t) = \begin{cases} 1 & t > 0 \\ 0 & t < 0 \end{cases}$$

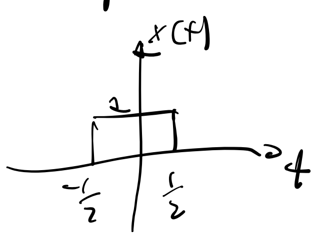
Esercizio 1

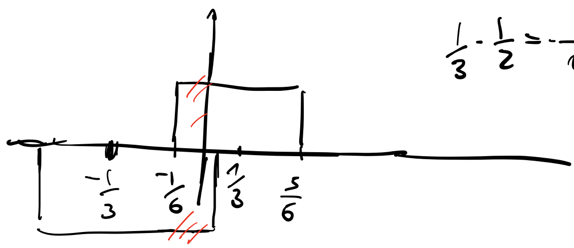
Dati

$$x(t) = \mathrm{rect}(t) \qquad y(t) = \mathrm{rect}\left(t-\tfrac{1}{3}\right) - \mathrm{rect}\left(t+\tfrac{1}{3}\right)$$

$$z(t) = x(t) + y(t)$$

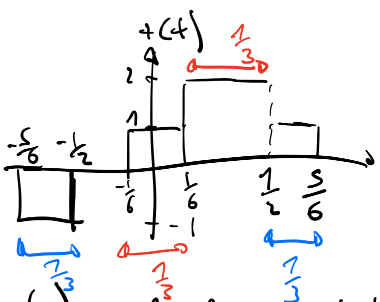
a) Disegnare $x(t)$, $y(t)$

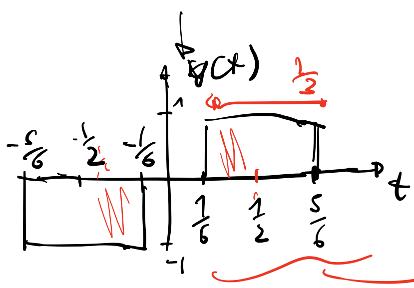




$\frac{1}{3} - \frac{1}{2} = -\frac{1}{6}$

$\frac{3}{2}\left(t-\frac{1}{2}\right)$





$$y(t) = \mathrm{rect}\left(\tfrac{3}{2}\left(t-\tfrac{1}{2}\right)\right) - \mathrm{rect}\left(\tfrac{3}{2}\left(t+\tfrac{1}{2}\right)\right)$$

b) Calcolare $\|x\|^2$, $\|y\|^2$ e $\langle x, y\rangle$

$\frac{2}{3} - \frac{1}{3}$

$$\|x\|^2 = \int_{-\frac{1}{2}}^{\frac{1}{2}} |1|^2 \, dt = 1 \qquad \|y\|^2 = 2\int_{\frac{1}{6}}^{\frac{5}{6}} |1|^2 \, dt$$

$$= 2 \cdot \frac{2}{3} \cdot 1 = \frac{4}{3}$$

$$\|y\|^2 = \|r_+ - r_-\|^2 = \underbrace{\|r_+\|^2}_{=1} + \underbrace{\|r_-\|^2}_{=1} - 2\underbrace{\langle r_+, r_- \rangle}_{\text{intersezione } \frac{1}{3}}$$

conti li evito perché $= 2 - \frac{1}{3} = \boxed{\frac{4}{3}}$

$$\langle x, y \rangle = \int_{-\frac{1}{2}}^{\frac{1}{2}} y(t) \, dt = \int_{-\frac{1}{2}}^{-\frac{1}{6}} (-1) \, dt + \int_{\frac{1}{6}}^{\frac{1}{2}} (1) \, dt$$

$$= 0$$

c) $x \perp y$ perché $\langle x, y \rangle = 0$

$$\|z\|^2 = \|x\|^2 + \|y\|^2 + \underbrace{2 \operatorname{Re}\{\langle x, y \rangle\}}_{=0}$$

$$\overset{\text{pitagora}}{=} 1 + \frac{4}{3} = \boxed{\frac{7}{3}}$$

d) $\|z\|^2 = 3 \cdot \frac{1}{3} \cdot 1^2 + \frac{1}{3} \cdot 2^2 = 1 + \frac{4}{3} = \boxed{\frac{7}{3}}$

Esercizio 2

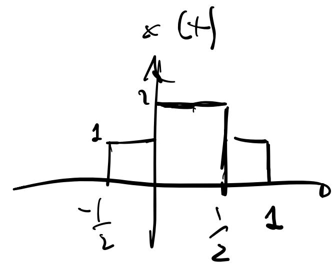
Def:

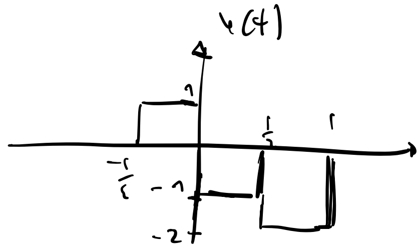
$$e(t) = \mathrm{rect}(t) \qquad b(t) = \mathrm{rect}\left(t - \frac{1}{2}\right)$$

si ha

$$x(t) = e(t) + b(t) \qquad y(t) = e(t) - 2b(t)$$

$$= \mathrm{rect}(t) + \mathrm{rect}\left(t-\frac{1}{2}\right) \qquad y(t) = \mathrm{rect}(t) - 2\,\mathrm{rect}\left(t-\frac{1}{2}\right)$$





a) $\|x\|^2 = \|e+b\|^2 = \|e\|^2 + \|b\|^2 + 2\langle e,b\rangle$

$$= 1 + 1 + 2\cdot\frac{1}{2} = 3$$

$\|y\|^2 = \|e - 2b\|^2 = \|e\|^2 + \|2b\|^2 - 2\langle e, 2b\rangle$

$$= \|e\|^2 + 4\|b\|^2 - 4\langle e,b\rangle$$

$$= 1 + 4 - 4\cdot\frac{1}{2} = 5 - 2 = 3$$

$\langle x,y\rangle = \langle e+b, e-2b\rangle = \langle e,e\rangle - 2\langle e,b\rangle + \langle b,e\rangle - 2\langle b,b\rangle$

$$= \underbrace{\langle e,e\rangle}_{\|e\|^2} - \langle e,b\rangle - 2\underbrace{\langle b,b\rangle}_{\|b\|^2}$$

$$= 1 - 2 - \frac{1}{2} = -1 - \frac{1}{2} = -\frac{3}{2}$$

b) Costruire base ortonormale $\{e_1, e_2\}$ di $\mathrm{span}\{x, y\}$ con $e_1 \parallel x$.

GRAM-SCHMIDT

$$e_1 = \frac{x}{\|x\|} = \frac{x}{\sqrt{3}} = \frac{1}{\sqrt{3}} \,\mathrm{vect}(t)$$

$$\tilde{e}_2 = \overbrace{y - \underbrace{\langle y, e_1\rangle e_1}_{(y_\parallel)}}^{y_\perp} = y + \frac{\sqrt{3}}{2} e_1 = y + \frac{1}{2} x$$

proiezione di y su e_1

$$\langle y, e_1\rangle = \langle y, \frac{1}{\sqrt{3}} x\rangle = \frac{1}{\sqrt{3}} \langle y, x\rangle = \frac{1}{\sqrt{3}} \langle x, y\rangle = \frac{1}{\sqrt{3}} \left(-\frac{3}{2}\right) = \boxed{-\frac{\sqrt{3}}{2}}$$

$$e_2 = \frac{\tilde{e}_2}{\|\tilde{e}_2\|} = \frac{2}{3} y + \frac{1}{3} x$$

$$\|\tilde{e}_2\|^2 = \|y + \tfrac{1}{2} x\|^2 = \|y\|^2 + \frac{1}{4}\|x\|^2 + 2 \underbrace{\langle y, \tfrac{1}{2} x\rangle}_{=\langle x, y\rangle}$$

$$\|\tilde{e}_2\| = \frac{3}{2} \qquad = 3 + \frac{1}{4} \cdot 3 + \left(-\frac{3}{2}\right) = 3 + \frac{3}{4} - \frac{3}{2} = 3 - \frac{3}{4} = \frac{9}{4}$$

c) coordinate x e y in $\{e_1, e_2\}$

coordinate su (e_1, e_2)

$$x \cong \left(\underbrace{\langle x, e_1\rangle}_{\sqrt{3}}, \underbrace{\langle x, e_2\rangle}_{0} \right) \equiv (\sqrt{3}, 0)$$

$$\langle y, \tfrac{2}{3} y + \tfrac{1}{3} x\rangle \quad \left(-\frac{3}{2}\right)$$

$$= \frac{2}{3} \underbrace{\|y\|^2}_{=3} + \frac{1}{3} \widetilde{\langle x, y\rangle}$$

$$= 2 \quad - \frac{1}{2} = \frac{3}{2}$$

$$y \cong \left(\langle y, e_1\rangle, \langle y, e_2\rangle \right) \equiv \left(-\frac{\sqrt{3}}{2}, \frac{3}{2}\right)$$

d) Esprimere a e b come combinazioni lineari di x e y e trovare le coordinate in (e_1, e_2).

$$\begin{cases} x = a + b \\ y = a - 2b \end{cases} \Rightarrow a = x - b \Rightarrow \begin{cases} a = x - \frac{x-y}{3} = \frac{3x - x + y}{3} = \frac{2x+y}{3} \\ y = x - b - 2b \Rightarrow b = \frac{x-y}{3} \end{cases}$$

$$a = \frac{2x+y}{3} \qquad b = \frac{x-y}{3}$$

$$a = \frac{1}{3}(2x+y) \qquad b = \frac{1}{3}(x-y)$$

$$(2\sqrt{3}, 0) + \left(-\frac{\sqrt{3}}{2}, \frac{3}{2}\right)$$

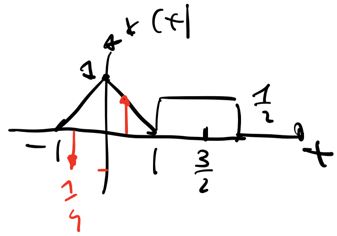
$$a = \frac{1}{3}\left[2(\sqrt{3}, 0) + \left(-\frac{\sqrt{3}}{2}, \frac{3}{2}\right)\right] = \frac{1}{3}\left(\frac{3}{2}\sqrt{3}, \frac{3}{2}\right) = \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$$

$$b = \frac{1}{3}\Big[\underbrace{(\sqrt{3}, 0) - \left(-\frac{\sqrt{3}}{2}, \frac{3}{2}\right)}\Big]$$

$$\frac{1}{3}\left(\frac{3}{2}\sqrt{3}, -\frac{3}{2}\right) = \left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$$

Esercizio 3

Sia $x(t) = \text{tri}(t) + \frac{1}{2}\,\text{rect}\left(t - \frac{3}{2}\right)$



a) Semplificare

$$y(t) = x(t)\,\delta\left(t - \frac{1}{2}\right) - 2\,x(t)\,\delta\left(t + \frac{3}{4}\right) + 3\,x(t)\,\delta\left(t - \frac{3}{2}\right)$$

$$y(t) = \underbrace{x\left(\frac{1}{2}\right)}_{=\frac{1}{2}}\delta\left(t - \frac{1}{2}\right) - \underbrace{2\,x\left(-\frac{3}{4}\right)}_{\underbrace{\frac{1}{4}(-2)}_{-\frac{1}{2}}}\delta\left(t + \frac{3}{4}\right) + \underbrace{3\,\underbrace{x\left(\frac{3}{2}\right)}_{\frac{1}{2}}}_{\frac{3}{2}}\delta\left(t - \frac{3}{2}\right)$$

$$= \frac{1}{2}\,\delta\left(t - \frac{1}{2}\right) - \frac{1}{2}\,\delta\left(t + \frac{3}{4}\right) + \frac{3}{2}\,\delta\left(t - \frac{3}{2}\right)$$

b) Calcola $\int_{-\infty}^{\infty} y(t)\,dt = \int_{-\infty}^{+\infty} \frac{1}{2}\,\delta\left(t - \frac{1}{2}\right) dt + \dots$

$$= \cancel{\frac{1}{2}} - \cancel{\frac{1}{2}} + \frac{3}{2} = \boxed{\frac{3}{2}}$$

c) Consideriamo il pettine di delta traslato

$$\delta_1\left(t-\frac{1}{2}\right)=\sum_{n\in\mathbb{Z}} \delta\left(t-\left(n+\frac{1}{2}\right)\right)$$

$$x_\delta(t)=x(t)\,\delta_1\left(t-\frac{1}{2}\right)$$

$$=\sum_{n\in\mathbb{Z}} x(t)\,\delta\left(t-\left(n+\frac{1}{2}\right)\right)$$

$$=\sum_{n\in\mathbb{Z}} x\left(n+\frac{1}{2}\right)\delta\left(t-\left(\frac{n+1}{2}\right)\right)$$

d)

$$x_\delta(t)=\frac{1}{2}\,\delta\left(t+\frac{1}{2}\right)+\frac{1}{2}\,\delta\left(t-\frac{1}{2}\right)+\frac{1}{2}\,\delta\left(t-\frac{3}{2}\right)$$

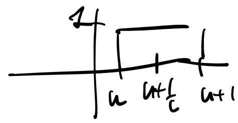
$$=\frac{1}{2}\left[\delta\left(t+\frac{1}{2}\right)+\delta\left(t-\frac{1}{2}\right)+\delta\left(t-\frac{3}{2}\right)\right]$$

$$\int_{-\infty}^{+\infty} x_\delta(t)\,dt=\frac{1}{2}\cdot 3=\frac{3}{2}$$

Esercizio 4

Per $n \in \mathbb{Z}$

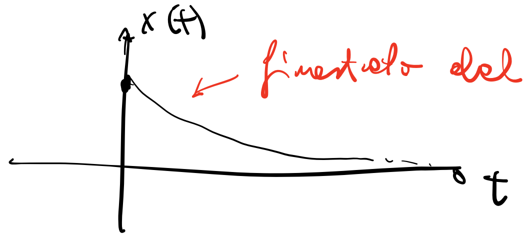
$$\varphi_n(t) = \mathrm{rect}\left(t - n - \tfrac{1}{2}\right) \qquad -\left(n+\tfrac{1}{2}\right)$$



quindi $\varphi_k(t) = 1$ per $t \in [n, n+1]$ 0 altrove.

Sia

$$x(t) = e^{-t}\,\varepsilon(t) \rightarrow \text{gradino}$$



finestrato del gradino

a) Mostrare che $\{\varphi_n\}_{n\in\mathbb{Z}}$ è insieme ortonormale

supporti disgiunti $\langle \varphi_{n_1}, \varphi_{n_2}\rangle = 0 \quad \forall\, n_1, n_2$ distinti

$$\|\varphi_n\|^2 = 1 \rightarrow \text{rect ampiezza 1 base 1}$$

b) Consideriamo $V = \overline{\mathrm{span}\{\varphi_n : n \geq 0\}}$ e prendiamo la proiezione

$$\hat{x}(t) = \sum_{n=0}^{\infty} c_n \varphi_n(t) \qquad c_n = \langle x, \varphi_n \rangle$$

base ortonormale di V.

$$c_n = \langle x, \varphi_n\rangle = \int_n^{n+1} e^{-t}\, dt$$

$$= -\left[e^{-t}\right]_n^{n+1} = -\left[e^{-(n+1)} - e^{-n}\right]$$

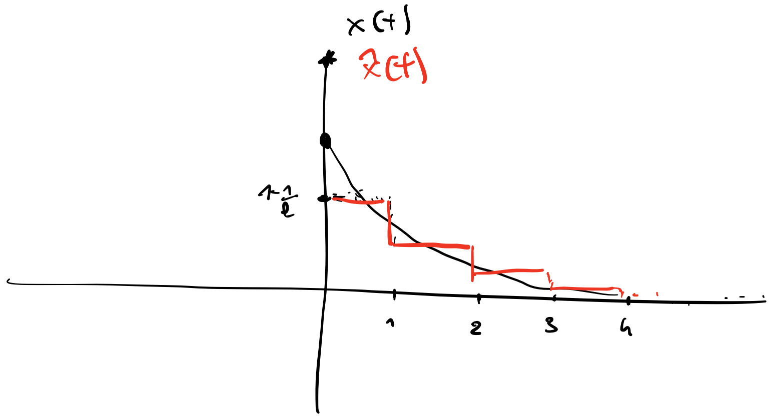
è il valore medio di e^{-t} su intervallo ☒.

$$= e^{-n} - e^{-n-1} = e^{-n}\left(1 - \tfrac{1}{e}\right)$$

c) $\underbrace{\|x\|^2}_{\text{energia}} = \int_0^{\infty} e^{-2t}\,dt = -\frac{1}{2}\left[e^{-2t}\right]_0^{\infty}$

$= -\frac{1}{2}\left[-1\right] = \frac{1}{2}$

$$\hat{x}(t) = \sum_{n=0}^{\infty} e^{-n}\left(1-\frac{1}{e}\right) \text{rect}\left(t-\left(n+\frac{1}{2}\right)\right)$$



$$\|\hat{x}(t)\|^2 = \sum_{n=0}^{\infty} \underbrace{\int_n^{n+1} \left[e^{-n}\left(1-\frac{1}{e}\right)\right]^2 dt}_{\text{não depende de } t}$$

$$\sum_{n=0}^{\infty} \left[e^{-n}\left(1-\frac{1}{e}\right)\right]^2 = \sum_{n=0}^{\infty} \left(1-\frac{1}{e}\right)^2 e^{-2n}$$

$$= \left(1-\frac{1}{e}\right)^2 \sum_{n=0}^{\infty} e^{-2n}$$

$$\sum_{n=0}^{\infty} \left(e^{-2}\right)^n \qquad \frac{1}{1-\frac{1}{e^2}} \qquad \frac{e^2}{e^2-1}$$

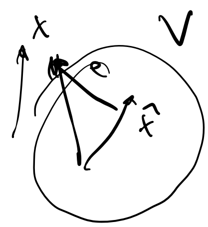
$$= \frac{(e-1)^2}{\cancel{e^2}} \cdot \frac{\cancel{e^2}}{e^2-1} = \frac{(e-1)^{\cancel{2}}}{\cancel{(e-1)}(e+1)} = \boxed{\frac{e-1}{e+1}} \approx 0.462$$

$$d(x,\hat{x})^2 = \|x-\hat{x}\|^2 = \|x\|^2 - \|\hat{x}\|^2$$

$$= \frac{1}{2} - \frac{e-1}{e+1}$$

$$= \frac{e+1-2e+2}{2(e+1)}$$

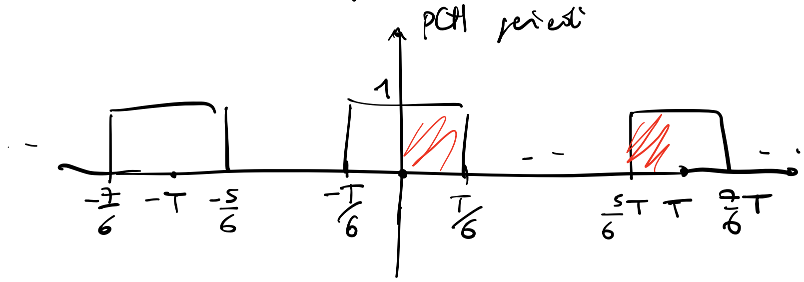
$$= \boxed{\frac{3-e}{2(e+1)}}$$



ESERCIZIO 5

Sia $p(t) = rect\left(\frac{3t}{T}\right)$ con $T > 0$

e periodizzato con periodo T $\quad rect\left(\frac{t}{T/3}\right)$



$\left[-\frac{T}{6}, \frac{T}{6}\right]$

Consideriamo $V = span\{1, \cos(2\pi t/T)\}$

costante $\quad$ periodo T del coseno

a) calcola $\bar{x} = \langle x, 1\rangle_T$, $P_x = \|x\|_T^2$

$$\bar{x} = \frac{1}{T}\int_0^T 1\cdot x(t)\,dt = \frac{1}{T}\cdot\left(\frac{T}{3}\right) = \frac{1}{3}$$

$$P_x = \frac{1}{T}\int_0^T |x(t)|^2 dt = \frac{1}{T}\left(1^2 \cdot \frac{T}{3}\right) = \frac{1}{3}$$

b)

$$\hat{x}(t) = \underbrace{\bar{x}}_{\frac{1}{3}} + A\cos(2\pi t/T)$$

cioè la proiezione di x su V

$\{1, \cos(2\pi t/T)\}$ base ortogonale

$1 \perp \cos(2\pi t/T)$

$$\int_0^T \cos(2\pi t/T) = 0$$ nel periodo

$$\langle \cos, \cos\rangle_T = \frac{1}{T}\int_0^T \cos^2(2\pi t/T)\, dt = \frac{1}{2}$$

$$A = \frac{\langle x, \cos\rangle_T}{\underbrace{\langle \cos, \cos\rangle_T}_{\frac{1}{2}}}$$

$$\frac{1}{T}\int_{-T/2}^{T/2} x(t)\cos(2\pi t/T)\, dt$$

$$= \frac{1}{T}\left[\int_{-T/6}^{T/6} \cos(2\pi t/T)\, dt\right]$$

$$\frac{1}{T}\left[\frac{T}{2\pi}\sin\left(2\pi\frac{t}{T}\right)\right]_{-T/6}^{T/6}$$

$$= \frac{\cancel{T}}{2\pi \cancel{T}}\left[2 \cdot \sin\left(\frac{\pi}{3}\right)\right] = \frac{\cancel{2}}{2\pi}\cdot\frac{\sqrt{3}}{\cancel{2}} = \frac{\sqrt{3}}{2\pi}$$

$$= \frac{\sqrt{3}}{2\pi} \cdot 2 = \frac{\sqrt{3}}{\pi}$$

$$P_{\hat{x}} = \|\hat{x}\|^2 = \bar{x}^2 + \frac{A^2}{2} = \left(\frac{1}{3}\right)^2 + \frac{3}{2\pi^2} = \frac{1}{9} + \frac{3}{2\pi^2}$$

Pot. cos.

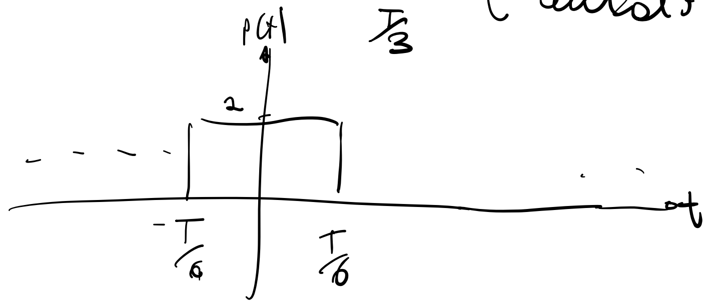
$$P_e = P_x - P_{\hat{x}} = \frac{1}{3} - \frac{1}{9} - \frac{3}{2\pi^2} = \frac{2}{9} - \frac{3}{2\pi^2}$$

ESERCIZIO 6

Fissato un $T > 0$ periodo. Per $n \in \mathbb{Z}$ definiamo

$$e_n(t) = e^{j2\pi \frac{n}{T} t}$$

consideriamo $p(t) = rect\left(\frac{3}{T} t\right)$ è centrato in 0 durata $\frac{T}{3}$



periodicizzato periodo T.

a) mostrare che $\{e_n\}_{n \in \mathbb{Z}}$ è insieme ortonormale

$$\langle e_m, e_n \rangle = \frac{1}{T} \int_0^T e^{j2\pi \frac{m}{T} t} \cdot e^{-j2\pi \frac{n}{T} t} \, dt$$

$$\frac{d}{dt}\left(e^{jat}\right) = \frac{d}{dt}\left(\cos(at) + j\sin(at)\right)$$

$ja\, e^{jat}$

$-a\sin(at) + ja\cos(at)$

$ja[-\sin(at) + \cos(at)]$

$e^{j2\pi \frac{t}{T}(m-n)}$

$\cos\left(\frac{(m-n)t}{N}\right) + j\sin(\cdot)$

$+j\sin(\omega t) + \cos(\omega t)$

$j e^{j\omega t}$

$m = n \Rightarrow 1$

$m \neq n$

$$\left[\frac{e^{j2\pi\frac{t}{T}(m-n)}}{\underbrace{j2\pi\frac{(m-n)}{T}}_{=0}}\right]_0^T$$

b) $V_1 = \mathrm{span}\{e_{-2}, e_0, e_2\}$

Calcolare c_0 e c_2 e c_{-2} dove $c_k = \langle x, e_k \rangle$

$$c_0 = \frac{1}{T}\int_{-T/6}^{T/6} 1\, dt = \frac{1}{T}\cdot\frac{T}{3} = \frac{1}{3}$$

$$c_k = \frac{1}{T}\int_{-T/6}^{T/6} e^{-j2\pi k\frac{t}{T}}\, dt$$

$$\frac{1}{\cancel{T}}\left[\cancel{T}\,\frac{e^{-j2\pi k\frac{t}{T}}}{-j2\pi k}\right]_{-T/6}^{T/6}$$

$$\frac{1}{6}\left[\underbrace{-\frac{e^{-j2\pi k\frac{1}{6}}}{j2\pi k/6} + \frac{e^{+j2\pi k\frac{1}{6}}}{j2\pi k/6}}_{\sin(2\pi k/6)}\right]$$

$\mathrm{sinc}(t)$

$\dfrac{\sin(\pi t)}{\pi t}$

$$\frac{1}{\cancel{6}}\cdot\frac{\cancel{6}}{\frac{\pi k}{3}\cdot 3}\left[\underbrace{\frac{e^{j2\pi k/6} - e^{-j2\pi k/6}}{2j}}_{\sin(2\pi k/6)}\right]$$

$\pi k/3$

$\sin(2\pi k/6)$

$$= \frac{1}{3}\,\mathrm{sinc}\left(\frac{k}{3}\right)$$

$$C_1 = + \frac{\sin(\pi/3)}{\pi} = + \frac{\sqrt{3}}{2\pi} \qquad C_{-2} = + \frac{\sin(-\frac{2}{3}\pi)}{-2\pi} = \frac{\sqrt{3}}{4\pi}$$

$C_1 = C_{-1}^*$

c)

$$\hat{x}(t) = \frac{1}{3} \underbrace{-\frac{\sqrt{3}}{2\pi}}_{C_{-1}} e^{j2\pi t/T} \underbrace{-\frac{\sqrt{3}}{2\pi}}_{C_1} e^{-j2\pi t/T}$$

$$-\frac{\sqrt{3}}{\pi}\left[\underbrace{\frac{e^{j2\pi t/T} + e^{-j2\pi t/T}}{2}}\right]$$

$$\hat{x}(t) = \frac{1}{3} + \frac{\sqrt{3}}{2\pi} e^{j2\pi t/T} + \frac{\sqrt{3}}{4\pi} e^{-2\pi t/T}$$

d)

$$P_x = \|x\|^2 = \frac{1}{T}\int_{-T/2}^{T/2} |x(t)|^2 \, dt = \frac{1}{1}\int_{-T/6}^{T/6} 1^2 \, dt = \boxed{\frac{1}{3}}$$

ortonormale

$$P_{\hat{x}} = \|\hat{x}\|^2 = |C_{-1}|^2 + |C_1|^2 + |C_0|^2 = \frac{1}{9} + \left(\frac{\sqrt{3}}{2\pi}\right)^2 + \left(\frac{\sqrt{3}}{4\pi}\right)^2$$

$$= \frac{1}{9} + \frac{3}{4\pi^2} + \frac{3}{16\pi^2}$$

$$= \frac{1}{9} + \frac{15}{16\pi^2}$$

$$P_e = P_x - P_{\hat{x}} = \frac{1}{3} - \frac{1}{9} - \frac{15}{16\pi^2} = \boxed{\frac{2}{9} - \frac{15}{16\pi^2}}$$