ESERCITAZIONE 2

NOTAZIONE

Dato un segnale $x(t)$

$$X(f) = \int_{-\infty}^{+\infty} x(t)\, e^{-j2\pi f t}\, dt$$

trasformata fourier continua

$$x(t) = \int_{-\infty}^{+\infty} X(f)\, e^{j2\pi f t}\, df$$

trasformata inversa

Dato un segnale $x(t)$ con periodo T

frequenza

$$x(t) = \sum_{n\in\mathbb{Z}} c_n\, e^{j2\pi \frac{n}{T} t} \quad , \quad c_n = \frac{1}{T}\int_0^T x(t)\, e^{-j2\pi \frac{n}{T} t}\, dt$$

Riscrivo $x(t) = \sum_{n\in\mathbb{Z}} p(t-nT)$ dove $p(t)$ è la singola replica

allora $\boxed{c_n = \frac{1}{T}\, P\left(\frac{n}{T}\right)}$ → utile negli esercizi

Esercizio 1

a) Calcolare TF di $\delta(t-t_0)$

$$\mathcal{F}\{\delta(t-t_0)\} = \int_{-\infty}^{+\infty} \delta(t-t_0)\, e^{-j2\pi f t}\, dt$$

$$= \boxed{e^{-j2\pi f t_0}}$$

$\boxed{t_0=0}$

$$x(t)=\delta(t) \longleftrightarrow X(f)=1 \quad \text{(costante)}$$

b) DUALITÀ

$$x(t) \longleftrightarrow X(f)$$

$$X(t) \longleftrightarrow x(-f)$$

$$x(u) = \int_{-\infty}^{+\infty} X(t)\, e^{j2\pi u t}\, dt$$

$$\downarrow$$

$$x(-f) = \int_{-\infty}^{+\infty} \underbrace{X(t)\, e^{-j2\pi f t}}_{\text{la trasformata di } X}\, dt$$

Usare le proprietà per dedurre che

$$x(t)=1 \longleftrightarrow X(f)=\delta(f)$$

dalla dualità otteniamo che $1 \longleftrightarrow \delta(-f)=\delta(f)$

Dedurre anche che $\mathcal{F}\{e^{j2\pi f_0 t}\} = \delta(f-f_0)$

Da (a) abbiamo che $\underline{\delta(t-t_0) \longleftrightarrow e^{-j2\pi f t_0}}$

$$\underbrace{e^{j2\pi f_0 t} = e^{-j2\pi(-f_0)t}}_{x(t)} \longleftrightarrow \underbrace{\delta(-f-(-f_0))}_{X(-f)} = \delta(f-f_0)$$

$$\int e^{j2\pi f_0 t}\, e^{-j2\pi f t}\, dt$$

$$= \int_{-\infty}^{+\infty} e^{-j2\pi(f-f_0)t}\, dt = \begin{cases} 1 & f=f_0 \\ 0 & \text{altrove} \end{cases} \rightarrow \underline{\underline{\delta(f-f_0)}}$$

Esercizio 2

Dato

$$x(t) = e^{-t}\, u(t)$$ gradino

1) TF $\rightarrow X(f) = \dfrac{1}{1+j2\pi f}$

a) Inversione temporale $\quad x(-t) \longleftrightarrow X(-f)$

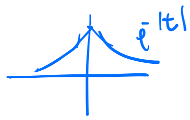
$$\underline{X(-f)} = \int_{-\infty}^{+\infty} x(t)\, e^{+j2\pi(+f)t}\, dt$$

$$\underline{\underline{x(-t)}}\, e^{-j2\pi f t}\, dt$$

b) Definire parte pari e dispari del segnale

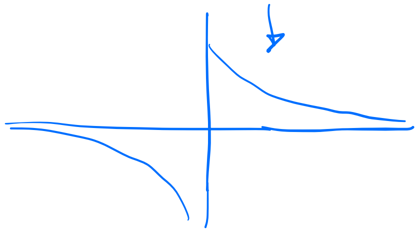
$$x(t) = \underbrace{x_e(t)}_{\text{pari}} + \underbrace{x_o(t)}_{\text{dispari}}$$

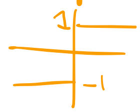
$$x_e(t) = \frac{x(t)+x(-t)}{2} \qquad x_o(t) = \frac{x(t)-x(-t)}{2}$$



$$x_e(t) = \frac{1}{2}\left[e^{-t}u(t) + e^{t}u(-t) \right] = \frac{1}{2} e^{-|t|}$$

$$x_o(t) = \frac{1}{2}\left[e^{-t} u(t) - e^{t} u(-t) \right] = \frac{1}{2} \underbrace{sgn(t)}_{\text{segno}} e^{-|t|}$$





c) Trovare $X_e(f)$ e $X_o(f)$ a partire da $X(f)$ e $X(-f)$

$$X_e(f) = \frac{X(f) + \overbrace{X(-f)}^{\text{inversione}}}{2} = \frac{1}{2}\left[\frac{1}{1+j2\pi f} + \frac{1}{1-j2\pi f} \right]$$

reale e pari

$$= \frac{1}{2}\left[\frac{2}{1+4\pi^2 f^2} \right] = \frac{1}{1+(2\pi f)^2}$$

$$X_o(f) = \frac{X(f) - X(-f)}{2} = \frac{1}{2}\left[\frac{1}{1+j2\pi f} - \frac{1}{1-j2\pi f} \right]$$

immaginaria e dispari

$$= \frac{1}{\cancel{2}}\left[\frac{-\cancel{2}j2\pi f}{1+(2\pi f)^2} \right] = -j \boxed{\frac{2\pi f}{1+(2\pi f)^2}}$$

Esercizio 3

Sia $x(t) = e^{-t} u(t)$, $X(f) = \frac{1}{1+j2\pi f}$

a) Traslazione

$y(t) = x(t - t_0)$ trovare $Y(f)$

$$Y(f) = \int_{-\infty}^{+\infty} x(\underbrace{t-t_0}_{t'=t-t_0})\, e^{-j2\pi f t}\, dt = \int_{-\infty}^{+\infty} x(t') \cdot e^{-j2\pi f (t'+t_0)}\, dt'$$

$$= e^{-j2\pi f t_0} \cdot X(f)$$

$$= e^{-j2\pi f t_0} \cdot \frac{1}{1+j2\pi f}$$

b)

$$z(t) = e^{j2\pi f_0 t} \cdot x(t)$$

$$\downarrow$$

$$Z(f) = \underbrace{\int_{-\infty}^{+\infty} x(t)\, \underbrace{e^{j2\pi f_0 t}\, e^{-j2\pi f t}}_{e^{-j2\pi t(f-f_0)}}\, dt}_{\boxed{X(f-f_0)}}$$

$$= X(f-f_0) = \frac{1}{1+j2\pi(f-f_0)}$$

ESERCIZIO 4

a) Calcolare TF $x(t) = \cos(2\pi f_0 t)$ segnale reale pari

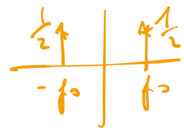
$$= \frac{e^{j2\pi f_0 t} + e^{-j2\pi f_0 t}}{2}$$

b) Calcolare TF $x(t) = \sin(2\pi f_0 t)$ segnale reale dispari

$$= \frac{e^{j2\pi f_0 t} - e^{-j2\pi f_0 t}}{2j}$$

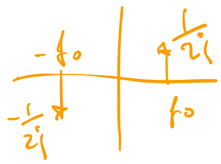
a) $X(f) = \frac{1}{2}\left[\underbrace{\mathcal{F}\{e^{j2\pi f_0 t}\}}_{\delta(f-f_0)} + \underbrace{\mathcal{F}\{e^{-j2\pi f_0 t}\}}_{\delta(f+f_0)}\right]$

$= \frac{1}{2}\left[\delta(f-f_0) + \delta(f+f_0)\right]$



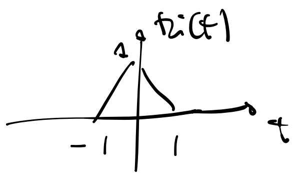
b) $X(f) = \frac{1}{2j}\left[\mathcal{F}\{e^{j2\pi f_0 t}\} - \mathcal{F}\{e^{-j2\pi f_0 t}\}\right]$

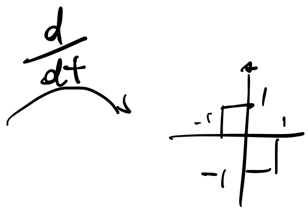
$= \frac{1}{2j}\left[\delta(f-f_0) - \delta(f+f_0)\right]$



Esercizio 5

Sia $x(t) = tri(t)$





a) Mostrare che $x'(t) = rect\left(t+\frac{1}{2}\right) - rect\left(t-\frac{1}{2}\right)$

b) Calcolare TF di $x'(t)$:

$rect(t) \leftrightarrow sinc(f)$

$\mathcal{F}\{x'(t)\} = e^{+j2\pi\frac{1}{2}f}\, sinc(f) - e^{-j2\pi\frac{1}{2}f}\, sinc(f)$

$= sinc(f)\left[\frac{e^{+j2\pi\frac{1}{2}f} - e^{-j2\pi\frac{1}{2}f}}{2j}\right]2j$

$= 2j\, sinc(f) \cdot \sin(\pi f)$

c) Dal punto precedente calcolare $X(f)$

$$x(t) = \underbrace{\int_{-\infty}^{+\infty} X(f)\, e^{j2\pi f t}\, df}_{\text{anti-trasformata}}$$

$$x'(t) = \frac{d}{dt} \int_{-\infty}^{+\infty} X(f)\, e^{j2\pi f t}\, df$$

sotto ipotesi di regolarità / Leibniz-rule

$$= \int_{-\infty}^{+\infty} \frac{d}{dt}\left(X(f)\, e^{j2\pi f t}\right) df$$

$$\underbrace{\int_{-\infty}^{+\infty} j2\pi f\, X(f)\, e^{j2\pi f t}\, df}_{\text{anti-trasformata}}$$

$$\boxed{x'(t) \longleftrightarrow j2\pi f\, X(f)}$$

$$\mathcal{F}\{x'(t)\} = \cancel{2j}\, \mathrm{sinc}(f)\, \sin(\pi f) = \cancel{j}2\pi f\, X(f)$$

$$\Downarrow$$

$$X(f) = \mathrm{sinc}(f)\, \frac{\sin(\pi f)}{\pi f} = \underline{\mathrm{sinc}^2(f)}$$

$= \mathrm{sinc}(f)$

$$-j2\pi t\, x(t) \longleftrightarrow X'(f)$$

$$X(f) = \int_{-\infty}^{+\infty} x(t)\, e^{-j2\pi f t}\, dt$$

$$\frac{d}{df} = \int_{-\infty}^{+\infty} -j2\pi t\, x(t)\, e^{-j2\pi f t}\, dt$$

Esercizio 6

Sia $x(t) = e^{-\pi t^2}$

a) Si vede che $x'(t) = -2\pi t \overbrace{\boxed{e^{-\pi t^2}}}^{x(t)} = -2\pi t \, x(t)$

$\Downarrow$ ODE

$$\boxed{2\pi t \, x(t) + x'(t) = 0} \quad *$$

b) Trasformiamo e troviamo un ODE per $X(f)$

$$\mathcal{F}\{2\pi t \, x(t)\} = -\frac{1}{j} \mathcal{F}\{-j 2\pi t \, x(t)\} = -\frac{1}{j} X'(f)$$
$$= j X'(f)$$

$$\mathcal{F}\{x'(t)\} = j 2\pi f \, X(f)$$

$$\boxed{\cancel{j} X'(f) + \cancel{j} 2\pi f \, X(f) = 0} \quad *$$

stesse

$X(0) = \int_{-\infty}^{+\infty} e^{-\pi t^2} = 1$ $\hookrightarrow$ $X(f) = e^{-\pi f^2}$

Esercizio 7

Trasformata del pettine di impulsi

$$\delta_T(t) = \sum_{n \in \mathbb{Z}} \delta(t - nT)$$

a) Trovare la serie di Fourier del pettine

$$\delta_T(t) = \sum_{k\in\mathbb{Z}} c_n \, e^{j2\pi \frac{k}{T} t} \qquad c_n = \frac{1}{T}\int_{-T/2}^{T/2} \delta(t)\cdot e^{-j2\pi \frac{k}{T} t}\, dt$$

$$= \frac{1}{T}\sum_{k\in\mathbb{Z}} e^{j2\pi \frac{k}{T} t} \qquad = \frac{1}{T}$$

$$\mathcal{F}\{\delta_T(t)\} = \frac{1}{T}\sum_{n\in\mathbb{Z}} \underbrace{\mathcal{F}\{e^{j2\pi \overset{\text{frequenze}}{\frac{k}{T}} t}\}}_{\delta(f-\frac{k}{T})}$$

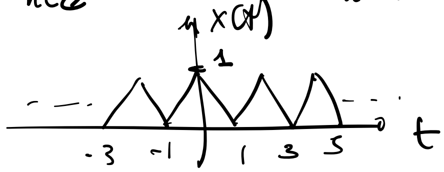
$$= \frac{1}{T}\sum_{n\in\mathbb{Z}} \delta\left(f-\frac{k}{T}\right) = \boxed{\frac{1}{T}\,\delta_{\frac{1}{T}}(f)}$$

Per $T=1$ abbiamo auto-similarità (come gaussiana).

Esercizio 8

Dato il seguente segnale periodico con $T=2$ con $p(t)=tri(t)$

$$x(t) = \sum_{n\in\mathbb{Z}} p(t-2n) = \sum_{n\in\mathbb{Z}} tri(t-2n)$$

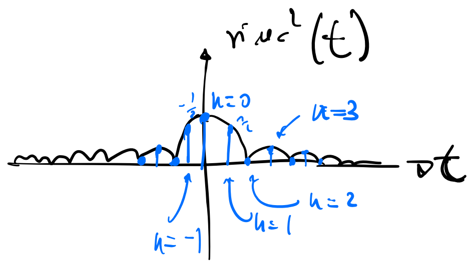


a) Usando $P(f)$ trovare i coefficienti della serie di Fourier di $x(t)$.

$$x(t) = \sum_{n\in\mathbb{Z}} c_n \, e^{j2\pi \frac{k}{2} t}$$

$P(f) = \text{sinc}^2(f)$

$C_n = \frac{1}{T} P\left(\frac{n}{T}\right) = \frac{1}{2} \underbrace{\text{sinc}^2\left(\frac{n}{2}\right)}_{\left[\frac{\sin\left(\pi \frac{n}{2}\right)}{\pi \frac{n}{2}}\right]^2}$



2 casi

n pari $C_n = 0$, n dispari $C_n = \frac{1}{2} \cdot \frac{4}{(\pi n)^2} = \frac{2}{\pi^2 n^2}$

$n \neq 0$

$C_0 = \frac{1}{2} \text{sinc}^2(0) = \frac{1}{2}$

$\sin\left(\pi \frac{n}{2}\right) = \pm 1$

$\text{sinc}\left(\frac{n}{2}\right) = \frac{\pm 2}{\pi n} \rightarrow \text{sinc}^2\left(\frac{n}{2}\right) = \frac{4}{(\pi n)^2}$

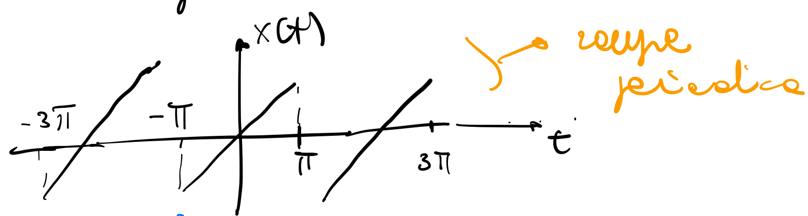
Siccome $x(t)$ è un segnale reale pari $\Rightarrow$ solo coseni

$x(t) = C_0 + 2 \sum_{n \text{ dispari}} \underbrace{\frac{2}{\pi^2 n^2}}_{C_n} \cdot \cos\left(\cancel{2}\pi \frac{n}{\cancel{2}} t\right)$

$\cos(\pi n t)$

ESERCIZIO 9

Sia $x(t)$ periodica con $T = 2\pi$



$x_p(t) = t$

a) Calcolare i coefficienti di Fourier data

$$X_p(f) = j\left[\frac{\cos(2\pi^2 f)}{f} - \frac{\sin(2\pi^2 f)}{2\pi^2 f^2}\right] \qquad f \neq 0$$

$$X_p(0) = 0$$

$$c_n = \frac{1}{2\pi} X_p\left(\frac{n}{2\pi}\right) = \frac{1}{2\pi} j \left[\frac{\cos\left(2\pi^2 \frac{n}{2\pi}\right)}{\frac{n}{2\pi}} - \frac{\overbrace{\cancel{\sin\left(2\pi^2 \frac{n}{2\pi}\right)}}^{=0}}{2\pi^2\left(\frac{n}{2\pi}\right)^2}\right]$$

$c_0 = 0$

$$= \frac{j}{\cancel{2\pi}} \cancel{2\pi} \frac{\cos(\pi n)}{n} = \boxed{\frac{j(-1)^n}{n}} \qquad n \neq 0$$

b) Calcola potenza $x(t)$ e $\sum |c_n|^2$

$$P_x = \frac{1}{2\pi}\int_{-\pi}^{\pi} t^2\, dt = \frac{1}{2\pi}\left[\frac{t^3}{3}\right]_{-\pi}^{\pi} = \frac{1}{\cancel{\pi}} \frac{\pi^3}{3} = \frac{\pi^2}{3}$$

$$\sum_{n\in\mathbb{Z}} |c_n|^2 = \sum_{n\in\mathbb{Z}\setminus\{0\}} \left|\frac{j(-1)^n}{n}\right|^2 = \sum_{n\in\mathbb{Z}\setminus\{0\}} \left|\frac{(-1)^n}{n}\right|^2 = \sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{1}{n^2}$$

$\Downarrow$

proiezione di $x(t)$ sulla base di fourier $\Rightarrow$ $P_x = \sum_{n\in\mathbb{Z}} |c_n|^2$

$$\frac{\pi^2}{3} = \sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{1}{n^2} = 2\sum_{n=1}^{\infty} \frac{1}{n^2} \Longrightarrow \boxed{\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}}$$