ESERCITAZIONE 3

NOTAZIONE

TF CONTINUA:

$$X(f) = \int_{-\infty}^{+\infty} x(t)\, e^{-j 2\pi f t}\, dt, \qquad x(t) = \int_{-\infty}^{+\infty} X(f)\, e^{j 2\pi f t}\, df$$

CONVOLUZIONE CONTINUA:

$$(x * y)(t) = \int_{-\infty}^{+\infty} x(\tau)\, h(t-\tau)\, d\tau$$

CONVOLUZIONE DISCRETA:

$$(x * y)[n] = \sum_{k=-\infty}^{\infty} x[k]\, y[n-k]$$

PARSEVAL:

$$\langle x, y \rangle = \int_{-\infty}^{+\infty} x(t)\, y^*(t)\, dt = \int_{-\infty}^{+\infty} X(f)\, Y^*(f)\, df = \langle X, Y \rangle$$

CASO UGUALE

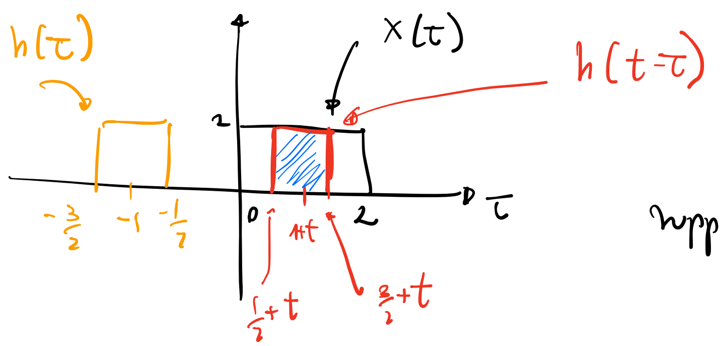
$\|x\|^2 = \|X\|^2$ Conservazione dell'energia.

Esercizio 1

Dato $x(t) = \text{rect}\left(\frac{t-1}{2}\right)$ e $h(t) = \text{rect}(t+1)$

calcolare $\underbrace{y(t) = (x * h)(t)}$

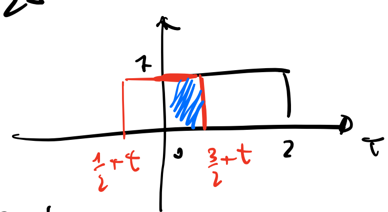
$$y(t) = \int_{-\infty}^{+\infty} x(\tau)\, h(t-\tau)\, d\tau$$



$\text{supp}(y) = \left[-\frac{3}{2}, \frac{3}{2}\right]$

- Fisso $x(\tau)$
- prendo $h(\tau)$ lo ribalto $h(-\tau)$
- traslo di t ottenendo $h(t-\tau)$
- Calcolo l'area di sovrapposizione

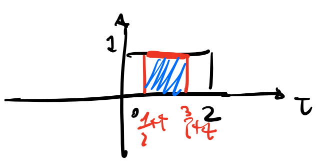
1° caso $t \in \left[-\frac{3}{2}, -\frac{1}{2}\right]$



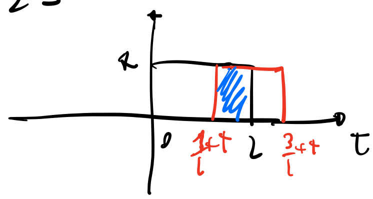
$y(t) =$ area $= \frac{3}{2}+t$

2° caso $t \in \left[-\frac{1}{2}, \frac{1}{2}\right]$

$y(t) =$ area $= 1$



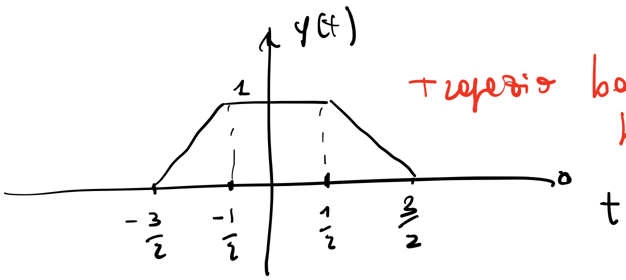
3° caso $t \in [\frac{1}{2}, \frac{3}{2}]$



$y(t) =$ area

$= 2 - (\frac{1}{2} + t) = \frac{3}{2} - t$

$\Downarrow$ $y(t)$



Trapezio base maggiore 3
base minore 1

Oss. Se eseguiamo anche le convoluzione di due rect con le stesse lunghezze allora in uscita avremo anche un tri.

$$rect(t) * rect(t) \xleftrightarrow{F} sinc^2(f)$$

$$\downarrow F^{-1}$$

$$tri(t)$$

$$rect(a(t-t_1)) * rect(a(t-t_2))$$

$$\downarrow F$$

$$\frac{1}{a^2} sinc^2(f/a) \cdot e^{-j\pi f(t_1+t_2)}$$

$$\downarrow F^{-1}$$

$$\frac{1}{a} tri(a(t-(t_1+t_2)))$$

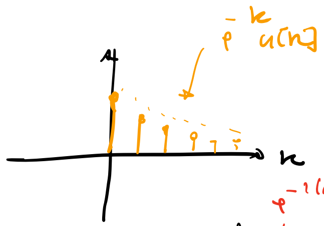
ESERCIZIO 2

Definiamo le sequenze

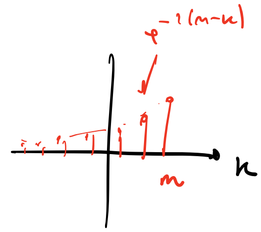
$$x[n] = e^{-n} u[n] \quad , \quad h[n] = e^{-2n} u[n]$$

dove $u[n]$ è il gradino, i.e., $u[n] = \begin{cases} 1 & n \geq 0 \\ 0 & \text{altrimenti} \end{cases}$

a) Calcolare $y[n] = (x * h)[n]$



$$y[n] = \sum_{k=-\infty}^{+\infty} x[k] \cdot h[n-k]$$



$$= \sum_{k=-\infty}^{\infty} e^{-k} u[k] \cdot e^{-2(n-k)} u[n-k]$$

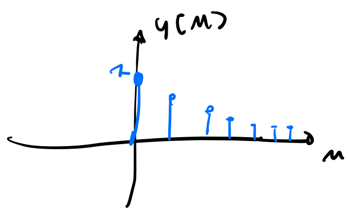
$$= \sum_{k=0}^{n} e^{-k} \cdot e^{-2n} \cdot e^{2k} = e^{-2n} \boxed{\sum_{k=0}^{n} e^{k}}$$

$$y[n] = e^{-2n} \cdot \boxed{\frac{e^{n+1} - 1}{e - 1}} = \frac{e^{-n+1} - e^{-2n}}{e - 1}$$

b) Calcolare i valori $y[0], y[1], y[2]$

$$y[0] = 1 \qquad y[1] = e^{-2}(1 + e) = e^{-1} + e^{-2}$$

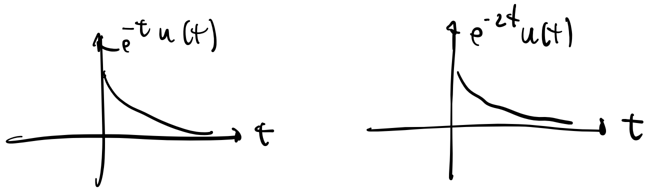
$$y[2] = e^{-4}\left[1 + e + e^{2}\right] = e^{-2} + e^{-3} + e^{-4}$$

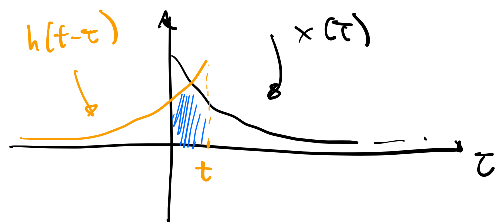


Esercizio 3

Dati $\quad x(t) = e^{-t} u(t) \quad , \quad h(t) = e^{-2t} u(t)$

a) Calcolare $y(t) = (x * h)(t)$ nel dominio del tempo





$$y(t) = \int_{-\infty}^{+\infty} x(\tau)\, h(t-\tau)\, d\tau = \int_{-\infty}^{+\infty} e^{-\tau} u(\tau)\, e^{-2(t-\tau)} u(t-\tau)\, d\tau$$

$$= \int_0^t e^{-\tau} e^{-2t} \cdot e^{2\tau}\, d\tau = e^{-2t} \int_0^t e^{\tau}\, d\tau$$

$$= e^{-2t} \left[e^{\tau} \right]_0^t = e^{-2t} \cdot \left[e^{t} - 1 \right]$$

$$y(t) = (e^{-t} - e^{-2t})\, u(t) \qquad = \boxed{e^{-t} - e^{-2t}} \qquad t \geq 0$$

b) Calcolare $X(f)$ e $H(f)$

Sapendo che $\quad p(t) = e^{-t} u(t) \longleftrightarrow \frac{1}{1 + j2\pi f} = P(f)$

$$p(2t) = e^{-2t} \underbrace{u(2t)}_{u(t)} = e^{-2t} u(t) = h(t)$$

$$p(2t) \longleftrightarrow \frac{1}{2} P\left(\frac{f}{2}\right) = \frac{1}{2} \cdot \frac{1}{1 + j2\pi \frac{f}{2}} = \frac{1}{\cancel{2}} \frac{\cancel{2}}{2 + j2\pi f} = \frac{1}{2 + j2\pi f}$$

$$X(f) = \frac{1}{1+j2\pi f}, \quad H(f) = \frac{1}{2+j2\pi f}$$

fratti semplici

$$Y(f) = X(f)H(f) = \frac{1}{1+j2\pi f} \cdot \frac{1}{2+j2\pi f} = \overbrace{\frac{A}{1+j2\pi f} + \frac{B}{2+j2\pi f}}$$

$$A(2+j2\pi f) + B(1+j2\pi f) = (2A+B) + j2\pi f(A+B)$$

$$\begin{cases} 2A + B = 1 & \rightarrow A = 1 \\ A+B=0 & \rightarrow B = -A = -1 \end{cases}$$

$$Y(f) = \frac{1}{1+j2\pi f} - \frac{1}{2+j2\pi f} \xrightarrow{\mathcal{F}^{-1}} \boxed{y(t) = \left(e^{-t} - e^{-2t}\right)u(t)}$$

Esercizio 4

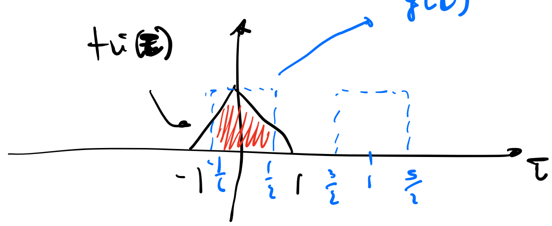
$$x(t) = \mathrm{tri}(t), \quad f_1(t) = \mathrm{rect}(t), \quad f_2(t) = \mathrm{rect}(t-2)$$

$$f(t) = f_1(t) + f_2(t)$$

a) Sia $y(t) = (x * f)(t)$, scriverla come convoluzioni semplici

$$y(t) = \underbrace{(x * f_1)(t)}_{} + \underbrace{(x * f_2)(t)}_{}$$

b) Calcolare $y(0)$ e $y(1)$



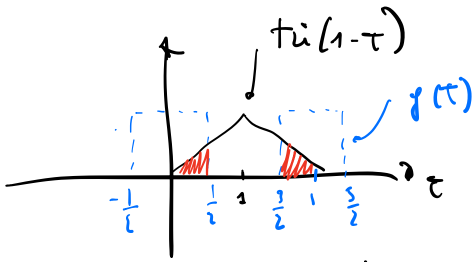
$$y(0) = \underbrace{(x * g_1)(0)} + \underbrace{(x * g_2)(0)}_{=0} = \frac{3}{4}$$

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \mathrm{tri}(\tau)\, d\tau = 2 \cdot \int_0^{\frac{1}{2}} (1-\tau)\, d\tau = 2 \cdot \left[\tau - \frac{\tau^2}{2}\right]_0^{\frac{1}{2}}$$

$$= 2 \cdot \left[\frac{1}{2} - \frac{1}{4} \cdot \frac{1}{2}\right]$$

$$= 2 \cdot \left[\frac{1}{2} - \frac{1}{8}\right] = 2 \cdot \frac{3}{8} = \boxed{\frac{3}{4}}$$

$$y(1) = (x * g_1)(1) + (x * g_2)(1)$$



$$2 \cdot //// = 2 \cdot \int_0^{\frac{1}{2}} (1-\tau)\, d\tau = \cancel{2} \cdot \underbrace{\frac{1}{2}}_{\text{base}} \cdot \underbrace{\frac{1}{2} \cdot \frac{1}{\cancel{2}}}_{\text{altezza}} = \boxed{\frac{1}{4}}$$

area triangolo

Esercizio 5

a) Utilizzare Parseval per mostrare che

$$\langle \underbrace{\mathrm{sinc}^2(t)}_{x(t)}, \underbrace{\mathrm{sinc}(t)}_{y(t)} \rangle = \int_{-\infty}^{+\infty} \mathrm{sinc}^3(t)\, dt = \int_{-\infty}^{+\infty} \mathrm{rect}(f)\, \mathrm{tri}(f)\, df$$

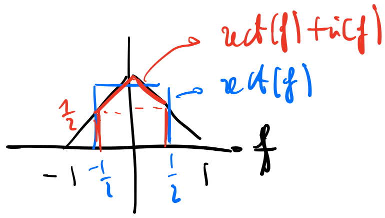
$$\mathrm{sinc}^2(t) \xleftrightarrow{\mathcal{F}} \mathrm{tri}(f)$$

$$\mathrm{sinc}(t) \xleftrightarrow{\mathcal{F}} \mathrm{rect}(f)$$

PARSEVAL $\langle x, y \rangle = \langle X, Y \rangle$

b) Calcolare $\int_{-\infty}^{+\infty} \mathrm{sinc}^3(t)\, dt$

$$= \int_{-\infty}^{+\infty} \mathrm{rect}(f)\,\mathrm{tri}(f)\, df = \int_{-\frac{1}{2}}^{\frac{1}{2}} \mathrm{tri}(f)\, df$$



$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} 1 - |f|\, df$$

$$= 2 \int_0^{\frac{1}{2}} (1-f)\, df = 2 \left[f - \frac{f^2}{2} \right]_0^{\frac{1}{2}}$$

$$= 2 \cdot \left[\frac{1}{2} - \frac{1}{8} \right] = \boxed{\frac{3}{4}}$$

c) Scrivere esplicitamente $z(f) = \mathrm{rect}(f) \cdot \mathrm{tri}(f)$

$$z(f) = \begin{cases} \boxed{1 - |f|} & |f| \le \frac{1}{2} \\ 0 & \text{altrimenti} \end{cases}$$

$|f| \le \frac{1}{2}$

$$z(f) = \frac{1}{2}\,\mathrm{rect}(f) + \frac{1}{2}\,\mathrm{tri}(2f)$$

$$= \frac{1}{2} + \frac{1}{2}\left(1 - 2|f|\right) = \boxed{1 - |f|}$$

d) Dedurre che $\mathcal{F}\{ \overbrace{\mathrm{sinc}^2(t) * \mathrm{sinc}(t)}^{z(t)} \} = Z(f)$

$= \mathrm{rect}(f)\,\mathrm{tri}(f)$

proprietà convoluzione

e mostrare quindi

che $z(t)$ è somme di due sinc/sinc^2.

$$Z(f) = \frac{1}{2}\,\mathrm{rect}(f) + \frac{1}{2}\,\mathrm{tri}(2f) \xleftrightarrow{\mathcal{F}^{-1}} z(t) = \frac{1}{2}\,\mathrm{sinc}(t) + \frac{1}{4}\,\mathrm{sinc}^2\!\left(\frac{t}{2}\right)$$