ESERCITAZIONE 4

ESERCIZIO 1

Fissato $T > 0$, definiamo

$$\phi_n(t) = \frac{1}{\sqrt{T}} \operatorname{sinc}\left(\frac{t - nT}{T}\right)$$

Mostrare che $\{\phi_n(t)\}$ è insieme ortonormale $L^2(\mathbb{R})$.

$$\langle \phi_n, \phi_m \rangle = \int_{-\infty}^{+\infty} \phi_n(t)\, \phi_m^*(t)\, dt$$

$$\overset{\text{PARSEVAL}}{=} \int_{-\infty}^{+\infty} \Phi_n(f)\, \Phi_m^*(f)\, df = \langle \Phi_n, \Phi_m \rangle$$

$$\Phi_n(f) = \frac{1}{\sqrt{T}} \cdot T \operatorname{rect}(Tf)\, e^{-j2\pi nTf} = \sqrt{T} \operatorname{rect}(Tf)\, e^{-j2\pi nTf}$$

$$* \quad \langle \Phi_n, \Phi_m \rangle = T \int_{-\frac{1}{2T}}^{\frac{1}{2T}} e^{-j2\pi nTf}\, e^{j2\pi mTf}\, df$$

$$= T \int_{-\frac{1}{2T}}^{\frac{1}{2T}} e^{-j2\pi T(n-m)f}\, df$$

$$= \operatorname{sinc}(n-m) = \begin{cases} 1 & n = m \\ 0 & n \neq m \end{cases}$$

ort. esponenziali complessi

$$B_T = \{ x(t) : X(f) = 0 \text{ per } |f| > \frac{1}{2T} \}$$

allora $\{\phi_n\}_{n\in\mathbb{Z}}$ è base ortonormale per B_T.

$x(t)$ è banda limitata $[-\frac{1}{2T}, \frac{1}{2T}]$

$$x(t) = \sum_{n\in\mathbb{Z}} c_n \, \phi_n(t) \qquad \text{dove} \qquad c_n = \langle x, \phi_n \rangle$$

valuto

$$x(mT) = \sum_{n\in\mathbb{Z}} c_n \, \underbrace{\phi_n(mT)}_{\frac{1}{\sqrt{T}} \operatorname{sinc}\left(\underbrace{\frac{T(m-n)}{T}}\right)}$$

$$\operatorname{sinc}(m-n) = \frac{\sin(\pi(m-n))}{\pi(m-n)}$$

$= 1$ $m = n$ $\qquad$ 0 altrimenti

$$\boxed{x(mT) = \frac{1}{\sqrt{T}} c_m}$$

$$x(t) = \sum_{n\in\mathbb{Z}} \sqrt{T}\, x(nT)\, \phi_n(t)$$

Esercizio 2

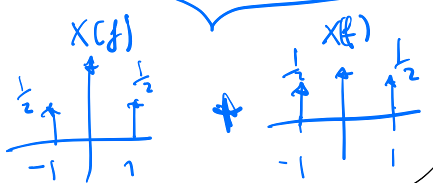
Prendiamo $x(t) = \cos(2\pi t)$, $y(t) = \cos(4\pi t)$

chiama $z_1(t) = x^2(t)$, $z_2(t) = x(t)\,y(t)$

a) Calcola $Z_1(f)$ e $Z_2(f)$

$$X(f) = \frac{1}{2}\left[\delta(f-1) + \delta(f+1)\right] \qquad Y(f) = \frac{1}{2}\left[\delta(f-2) + \delta(f+2)\right]$$

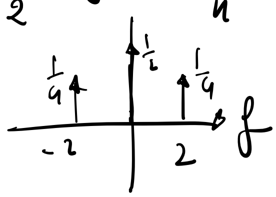
$$Z_1(f) = X(f) * X(f)$$



$$= \frac{1}{4}\left[\delta(f-1) + \delta(f+1)\right] * \left[\delta(f-1) + \delta(f+1)\right]$$

$$= \frac{1}{4}\left[\delta(f-2) + \delta(f+2) + \underbrace{\delta(f) + \delta(f)}_{2\,\delta(f)}\right]$$

$$= \frac{1}{2}\delta(f) + \frac{1}{4}\delta(f-2) + \frac{1}{4}\delta(f+2)$$

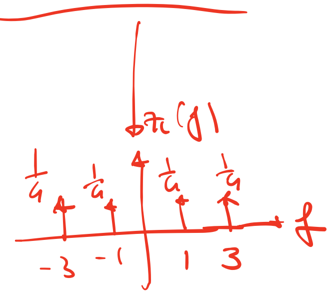


$$\xrightarrow{\mathcal{F}^{-1}} \frac{1}{2} + \frac{1}{2}\cos(4\pi t)$$

$$Z_2(f) = X(f) * Y(f)$$

$$= \frac{1}{4}\left[\delta(f-1) + \delta(f+1)\right] * \left[\delta(f-2) + \delta(f+2)\right]$$

$$= \frac{1}{4}\left[\delta(f-3) + \delta(f-1) + \delta(f+1) + \delta(f+3)\right]$$



$$\xrightarrow{\mathcal{F}^{-1}} \frac{1}{2}\cos(2\pi t) + \frac{1}{2}\cos(6\pi t)$$

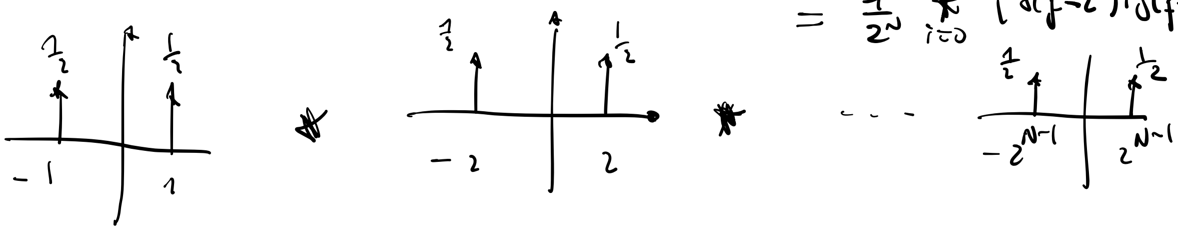
Esercizio 3

Consideriamo il segnale

$$x_N(t) = \prod_{i=0}^{N-1} \cos(2\pi\, 2^i t)$$

a) Calcolare $X_N(f) = \mathcal{F}\{x_N(t)\} = \mathop{\Large *}_{i=0}^{N-1} \frac{1}{2}\left[\delta(f-2^i)+\delta(f+2^i)\right]$

$$= \frac{1}{2^N} \mathop{\Large *}_{i=0}^{N-1} \left[\delta(f-2^i)+\delta(f+2^i)\right]$$



N volte

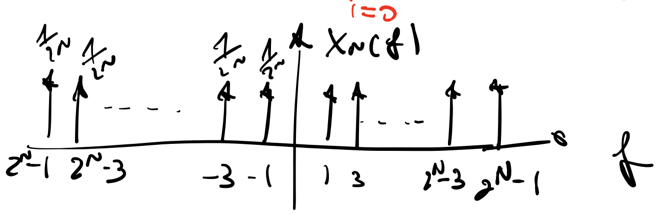
b) le frequenze che compaiono in uscita sono date da tutte le 2^N combinazioni di:

$$\underbrace{\pm 2^0 \pm 2^1 \pm 2^2 \pm \dots \pm 2^{N-1}}$$

Ogni scelta dei segni produce un intero distinto dispari $\Rightarrow$ abbiamo le delte sugli interi positivi dispari tra -2^N+1 e $\underbrace{2^N-1}$.

$$\sum_{i=0}^{N-1} 2^i = 2^N - 1$$

equiprob uniforme $\frac{1}{2^N}$



c) $$X_N(f) = \frac{1}{2^N} \sum_{\substack{m=-2^N+1 \\ m \text{ dispari}}}^{2^N-1} \delta(f-m)$$

$$= \frac{1}{2^N} \sum_{m=0}^{2^{N-1}-1} \left[\delta(f-(2m+1)) + \delta[f+(2m+1)] \right]$$

$$= \frac{1}{2^{N-1}} \sum_{m=0}^{2^{N-1}-1} \left[\frac{\delta(f-(2m+1)) + \delta(f+(2m+1))}{2} \right]$$

$\mathcal{F}^{-1}$

$$x_N(t) = \frac{1}{2^{N-1}} \sum_{m=0}^{2^{N-1}-1} \cos(2\pi(2m+1)t)$$

Esercizio 4

Dati $x(t) = e^{-\pi t^2}$ e $y(t) = e^{-2\pi t^2}$

Consideriamo

$$z(t) = x(t) * y(t)$$

a) Calcolare $X(f)$ e $Y(f)$

$$x(t) = e^{-\pi t^2} \longleftrightarrow e^{-\pi f^2} = X(f) \quad \text{(TRASF. NOTA)}$$

$$y(t) = x(\sqrt{2}\,t) \Rightarrow Y(f) = \frac{1}{\sqrt{2}} X\left(\frac{f}{\sqrt{2}}\right)$$

$$= \frac{1}{\sqrt{2}} \cdot e^{-\pi \frac{f^2}{2}}$$

b) Calcolare $Z(f)$

$$Z(f) = X(f)\,Y(f) = \frac{1}{\sqrt{2}} \cdot e^{-\pi f^2} \cdot e^{-\pi \frac{f^2}{2}}$$

$$= \frac{1}{\sqrt{2}}\, e^{-\pi f^2 \underbrace{\left(1+\frac{1}{2}\right)}_{3/2}}$$

$$= \frac{1}{\sqrt{2}}\, e^{-\frac{3}{2}\pi f^2}$$

$$Z(f) = \frac{1}{\sqrt{2}}\, X\left(\sqrt{\tfrac{3}{2}}\, f\right)$$

c) Antitrasformare $Z(f)$ e verificare che è ancora una gaussiana.

$$Z(f) = \frac{1}{\sqrt{2}}\, X\left(\sqrt{\tfrac{3}{2}}\, f\right) \longleftrightarrow \frac{1}{\sqrt{2}} \cdot \sqrt{\tfrac{2}{3}}\; x\left(\sqrt{\tfrac{2}{3}}\, t\right) = z(t)$$

$$\boxed{z(t) = \frac{1}{\sqrt{3}}\, e^{-\frac{2}{3}\pi t^2}} = \boxed{e^{-\pi t^2} * e^{-2\pi t^2}}$$

Esercizio 5

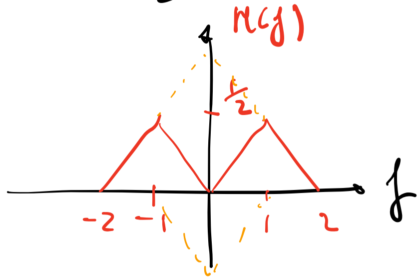
Dato un LTI con risposta all'impulso

$$h(t) = \mathrm{sinc}^2(t)\cos(2\pi t)$$

a) Calcolare la risposta in frequenza $H(f)$.

$$H(f) = \mathrm{tri}(f) * \left[\frac{1}{2}\delta(f-1) + \frac{1}{2}\delta(f+1)\right]$$

$$= \frac{1}{2}\mathrm{tri}(f-1) + \frac{1}{2}\mathrm{tri}(f+1)$$



$$H(f) = \frac{1}{2}\mathrm{tri}(f-1) + \frac{1}{2}\mathrm{tri}(f+1)$$

$$\Big\downarrow \mathcal{F}^{-1} \quad = \mathrm{tri}(f/2) - \mathrm{tri}(f)$$

$$h(t) = 2\,\mathrm{sinc}^2(2t) - \mathrm{sinc}^2(t) = \mathrm{sinc}^2(t)\cos(2\pi t)$$

b) Sistema è stabile?

$$\text{STABILE} \iff \underbrace{\int_{-\infty}^{+\infty} |h(t)|}_{\text{finito (integrabile assolutamente)}} < \infty$$

$$\int_{-\infty}^{+\infty} |\operatorname{sinc}^2(t)\cos(2\pi t)|\,dt = \int_{-\infty}^{+\infty} |\operatorname{sinc}^2(t)|\,\underbrace{|\cos(2\pi t)|}_{\leq 1}\,dt$$

$$= \int_{-\infty}^{\infty} |\operatorname{sinc}^2(t)|\,dt = \operatorname{tri}(0) = \boxed{1} < \infty$$

$\boxed{\text{STABILE}}$

c) Sistema è causale? No perchè $h(t)$ non è nullo per $t<0$.

d) Dato in ingresso

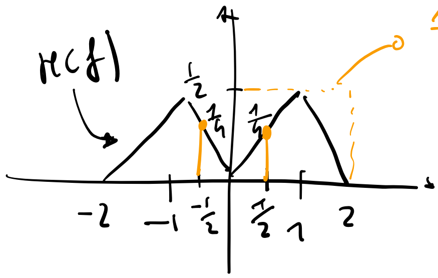
$$x(t) = \underbrace{\operatorname{sinc}(2t)\,e^{j\pi t}}_{x_1(t)} + \underbrace{\cos(\pi t)}_{x_2(t)} + \underbrace{\sin(6\pi t)}_{x_3(t)}$$

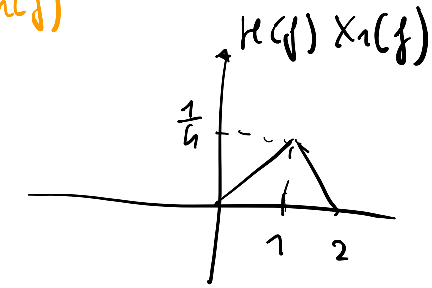
$$y(t) = S[x](t) = (x_1 * h)(t) + (x_2 * h)(t) + (x_3 * h)(t)$$

$$Y(f) = \overbrace{\boxed{X_1(f)}\, H(f)}^{y_1(t)} + \overbrace{X_2(f)\, H(f)}^{y_2(t)} + \overbrace{X_3(f)\, H(f)}^{y_3(t)}$$

$$X_1(f) = \frac{1}{2}\,\text{rect}\left(\frac{f-1}{2}\right)$$

$\frac{1}{2}\,\text{rect}\left(\frac{f-1}{2}\right) = X_1(f)$





$$Y_1(f) = X_1(f)\, H(f) \xrightarrow{\mathcal{F}^{-1}} \boxed{y_1(t) = \frac{1}{4}\,\text{sinc}^2(t)\, e^{j2\pi t}}$$

$$= \frac{1}{4}\,\text{tri}(f-1)$$

Teorema della risposta in frequenza:

$$y_2(t) = \underbrace{\left|H\left(\tfrac{1}{2}\right)\right|}_{=\frac{1}{4}} \cos\left(2\pi \cdot \frac{1}{2}\, t + \underbrace{\arg H\left(\tfrac{1}{2}\right)}_{=0}\right)$$

$$= \frac{1}{4}\cos(\pi t)$$

$$y_3(t) = \left|\underbrace{H(3)}_{=0}\right| \sin\left(2\pi \cdot 3t + \arg H(3)\right) \quad (\text{supporto } H(f) \in [-2,2])$$

$$= 0$$

$$y(t) = y_1(t) + y_2(t) + y_3(t) = \boxed{\frac{1}{4}\,\text{sinc}^2(t)\, e^{j2\pi t} + \frac{1}{4}\cos(\pi t)}$$