Esercitazione 5

Notazione

Campionamento ideale con periodo T_s e $f_s = \frac{1}{T_s}$

$$x_s(t) = x(t)\,\delta_{T_s}(t) = x(t)\sum_{n\in\mathbb{Z}} \delta(t - T_s) = \sum_{n\in\mathbb{Z}} x(nT_s)\,\delta(t - nT_s)$$

$$\updownarrow \mathcal{F}$$

$$X_s(f) = X(f) * \frac{1}{T_s}\,\delta_{\frac{1}{T_s}}(f) = f_s\, X(f) * \delta_{f_s}(f)$$

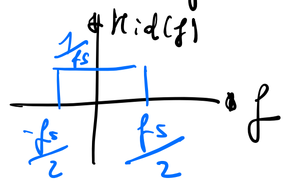
$$= f_s \sum_{k\in\mathbb{Z}} X(f) * \delta(f - k f_s)$$

$$= \underbrace{f_s \sum_{k\in\mathbb{Z}} X(f - k f_s)}_{\text{repliche spettrali}}$$

fs

Interpolatore ideale per ricostruzione perfetta (in assenza di aliasing $f_s > 2B$ dove B è la banda del segnale).

$$H_{id}(f) = T_s\, \mathrm{rect}(T_s f) = \frac{1}{f_s}\,\mathrm{rect}\left(\frac{f}{f_s}\right)$$



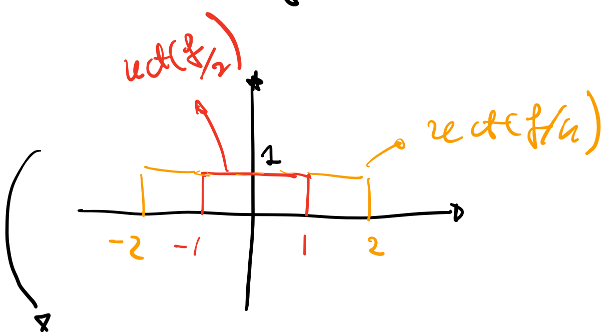
Esercizio 1

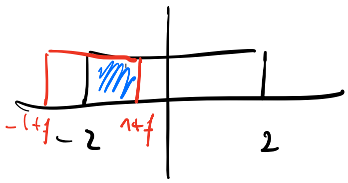
Si consideri $x(t) = \underbrace{\text{sinc}(2t)}_{\mathcal{F}\downarrow}\,\underbrace{\text{sinc}(4t)}_{\downarrow}$

a) Trovare $X(f)$

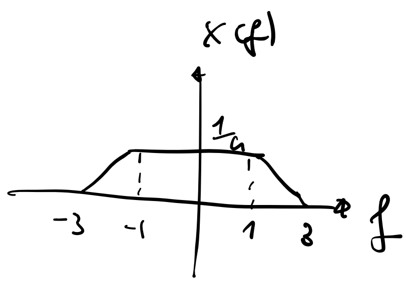
$$X(f) = \frac{1}{2}\,\text{rect}\left(\frac{f}{2}\right) * \frac{1}{4}\,\text{rect}(f/4)$$

$$= \frac{1}{8}\left[\text{rect}(f/2) * \text{rect}(f/4)\right]$$





$$X(f) = \begin{cases} 0 & f < -3 \\ \frac{3+f}{8} & -3 \leq f < -1 \\ \frac{2}{8} = \frac{1}{4} & -1 \leq f < 1 \\ \frac{3-f}{8} & 1 \leq f < 3 \\ 0 & f \geq 3 \end{cases} \longrightarrow$$



b/c) Determinare la frequenza minima di campionamento $f_{s,min}$ per una ricostruzione perfetta

$$f_s > 2B > 6$$

nel nostro caso $B=3$ (Banda)

Posso prendere $f_s = 2B$? In questo caso posso perché non ho salti/delta centrate in $\pm B$. Quindi

$\boxed{X(\pm 3) = 0} \longleftrightarrow f_{s,min} = 2B = \boxed{6}$.

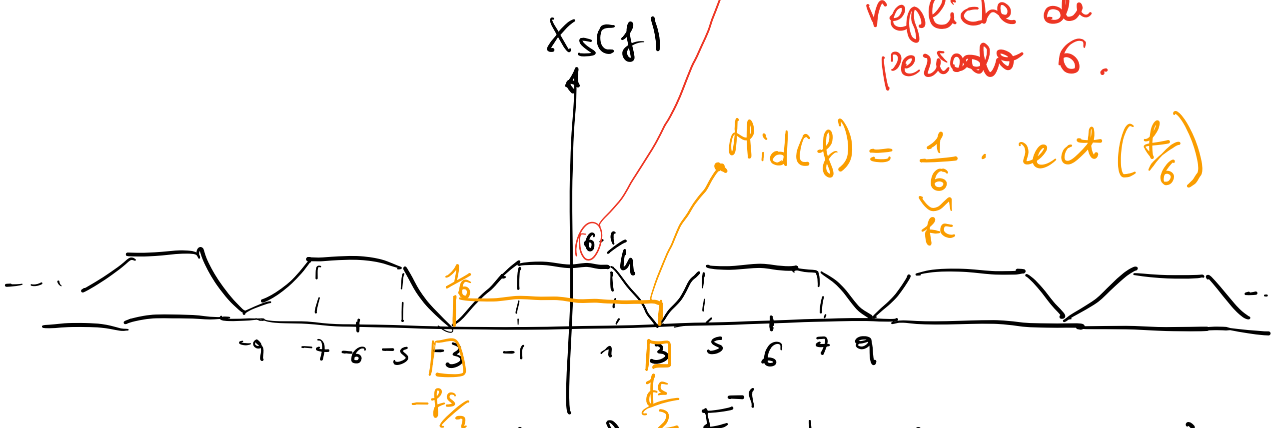
d) Nel caso $f_s = 6$, scrivere l'interpolatore ideale e la sua risposta impulsiva.

$$x_s(t) = x(t)\,\delta_{\frac{1}{6}}(t)$$

$\downarrow$

$$X_s(f) = X(f) * \left[6\,\delta_6(f)\right] = 6 \sum_{n \in \mathbb{Z}} X(f-6n)$$

repliche di periodo 6.



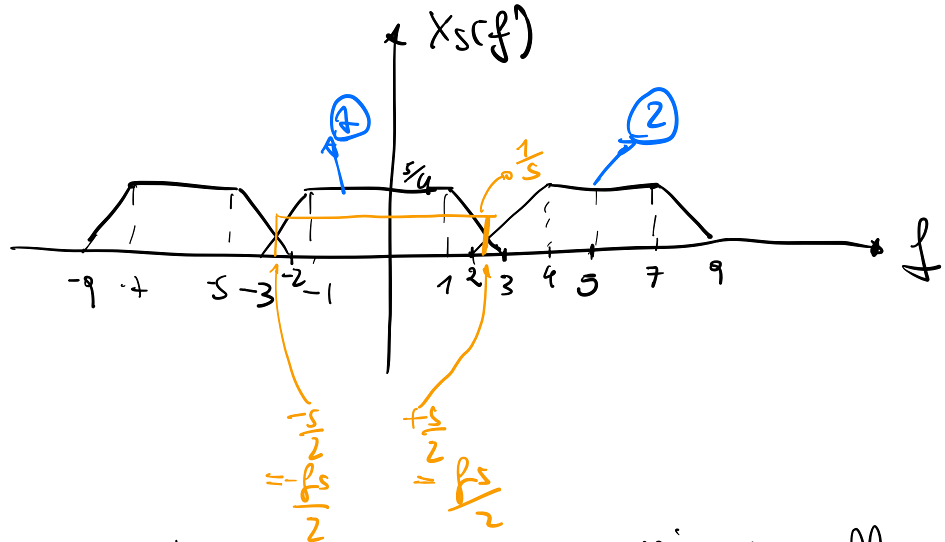
Quindi $H_{id}(f) = \frac{1}{6}\, rect(f/6) \xrightarrow{F^{-1}} h_{id}(t) = \frac{1}{\cancel{6}}\cancel{6} \cdot sinc(6t) = sinc(6t)$

e) Studiare cosa succede con $f_s = 5$.

$$5 = f_s < 2B = 6$$

Quindi c'è effetto di aliasing.

$$X_s(f) = 5 \underbrace{\sum_{n\in\mathbb{Z}} X(f-5n)}_{\text{repliche periodo } 5} \qquad H_{id}(f) = \frac{1}{5}\,\mathrm{rect}\left(\frac{f}{5}\right)$$

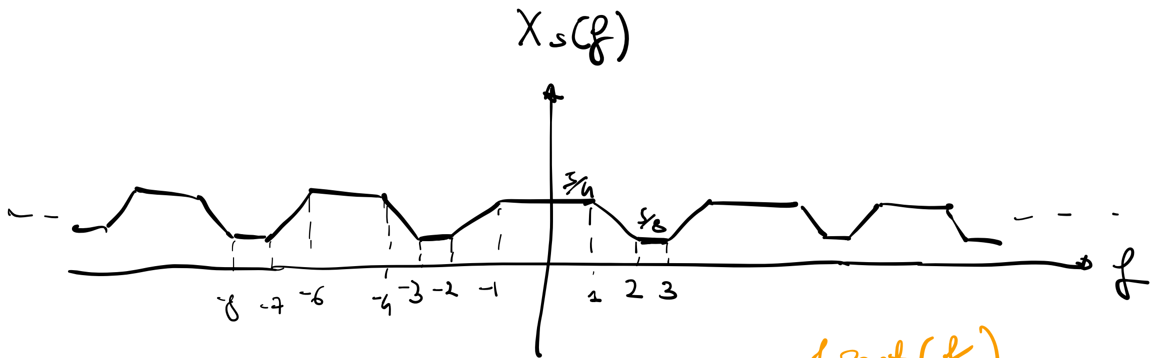


C'è sovrapposizione tra i due trapezi nell'intervallo $[-3,-2]$ e $[2,3]$.

① trapezio in $[2,3]$ vale $\frac{3-f}{8}$

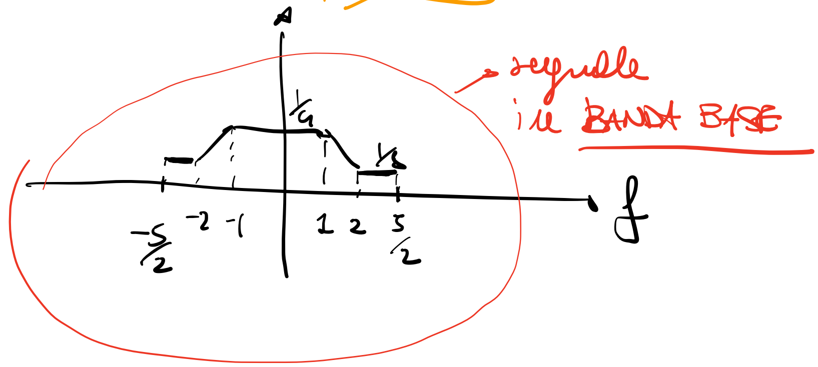
② trapezio in $[2,3]$ vale $\frac{3+(f-5)}{8}$

① + ② $\Longrightarrow$ $\frac{3-f}{8} + \frac{3+f-5}{8} = \boxed{\frac{1}{8}}$



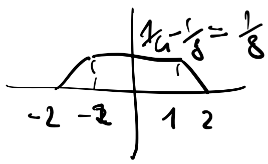
$$Y(f) = X_s(f)\cdot \boxed{H_{id}(f)}$$

$\frac{1}{5}\mathrm{rect}\left(\frac{f}{5}\right)$



Si può osservare che il segnale è costante in $[-\frac{5}{2}, 2]$ e $[2, \frac{5}{2}]$ ed è uguale ad $\frac{1}{8}$. Nell'intervallo $[-2, 2]$ è un trapezio elevato di $\frac{1}{8}$.

$Y(f) = \frac{1}{8}\mathrm{rect}\left(\frac{f}{5}\right) +$



$$\mathcal{F}^{-1}: \quad = \frac{1}{8}\mathrm{rect}\left(\frac{f}{5}\right) + \frac{1}{8}\left[\mathrm{rect}(f) * \mathrm{rect}\left(\frac{f}{3}\right)\right]$$

$$y(t) = \frac{5}{8}\,\mathrm{sinc}(5t) + \frac{3}{8}\,\mathrm{sinc}(t)\,\mathrm{sinc}(3t)$$

ESERCIZIO 2 (CAMPIONAMENTO CRITICO SINUSOIDE)

Sia $x(t) = A\cos(2\pi f_0 t + \phi)$, $A > 0$

Si campioni il segnale con $f_s = 2f_0$, $T_s = \frac{1}{2f_0}$

a) Calcolare i campioni

$$x[n] = x(nT_s)$$

$$x[n] = x(nT_s) = A\cos(2\pi f_0 n T_s + \phi)$$

$$= A\cos\left(\cancel{2}\pi \cancel{f_0} n \frac{1}{\cancel{2f_0}} + \phi\right)$$

$$(*) = A\cos(\pi n + \phi)$$

id. $\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$

b) Dipende solo da A e $\cos(\phi)$

$$(*) = A[\cos(\pi n)\cos(\phi) - \sin\underbrace{(\pi n)}_{=0}\sin(\phi)]$$

$$= A\underbrace{\cos(\pi n)}_{(-1)^n}\cos(\phi) = A(-1)^n\cos(\phi)$$

C) Discutere per quali valori di ϕ i campioni determinano il segnale originale. Fasi diverse possono produrre la stessa sequenza se prendo $-\phi$ ottengo $\cos(\phi) = \cos(-\phi)$

quindi $A\cos(\phi) = A\cos(-\phi)$

Dai campioni riesco a ricostruire $\cos(\phi)$ (quindi la cosinusoide) una volta fissato A. Ma non ϕ perché $\phi \in [0, 2\pi]$ e $\cos(\phi) = \cos(-\phi)$. $\phi = 2\pi - \phi \Rightarrow \boxed{\phi = \pi}$

$$A\cos(2\pi f_0 t + m\pi) \quad m \in \mathbb{Z}$$
$$= \pm A\cos(2\pi f_0 t)$$

<u>Es.</u>

$$\phi = \frac{\pi}{2} + m\pi \quad m \in \mathbb{Z} \quad \rightarrow \quad \boxed{\cos(\phi) = 0}$$

$x[n] = 0 \quad n \in \mathbb{Z}$ campioni <u>nulli</u>.

d) Studio nelle frequenze

$$x(t) = A\cos(2\pi f_0 t + \phi) = A\left[\frac{e^{j[2\pi f_0 t + \phi]} + e^{-j[2\pi f_0 t + \phi]}}{2}\right]$$

$$= \frac{A}{2}\left[\underbrace{e^{j\phi}} e^{j2\pi f_0 t} + \underbrace{e^{-j\phi}} e^{-j2\pi f_0 t}\right]$$

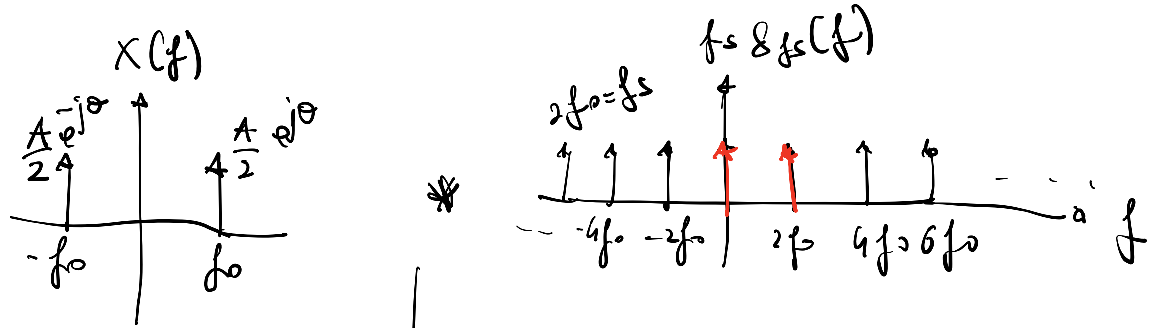
$\mathcal{F}$

non dipendono da t

$$X(f) = \frac{A}{2}\left[e^{j\phi}\delta(f - f_0) + e^{-j\phi}\delta(f + f_0)\right]$$

Campionamento ideale porta:

$$X_s(f) = \sum_{n\in\mathbb{Z}} X(f - n \underbrace{\boxed{f_s}}_{=2f_0})$$



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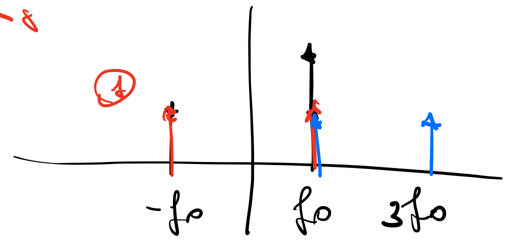
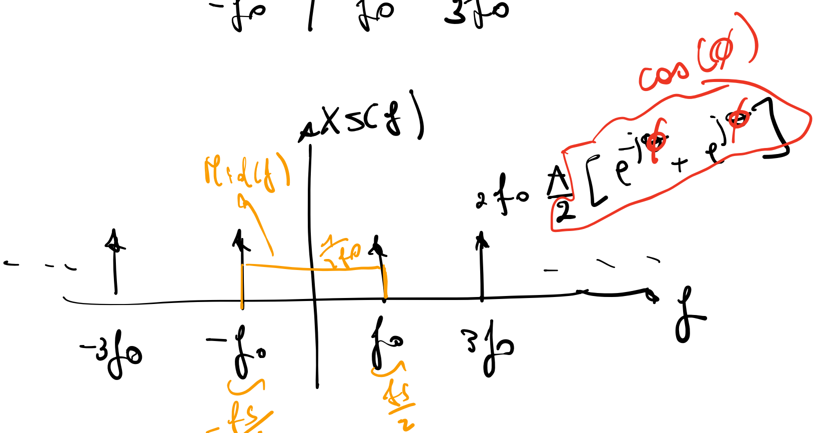
$$X(f) * 2f_0\,\delta(f) = 2f_0\,X(f)$$

$$X(f) * 2f_0\,\delta(f-2f_0) = 2f_0\,X(f-2f_0)$$

$$= 2f_0\left[\frac{A}{2}e^{-j\phi}\delta(f-f_0) + \frac{A}{2}e^{j\phi}\delta(f-3f_0)\right]$$





Ogni delta ha ampiezza $2f_0\,A\cos(\phi)$.

$$H_{id}(f) = \frac{1}{f_s}\, rect\left(\frac{f}{f_s}\right) = \frac{1}{2f_0}\, rect\left(\frac{f}{2f_0}\right)$$

osservando che $rect(\pm\frac{1}{2}) = \frac{1}{2}$

$$Y(f) = H_{id}(f)\, X_s(f) = \frac{1}{\cancel{2f_0}} \cdot \frac{1}{2} \cdot \cancel{2f_0} \cdot A\cos(\phi)\left[\delta(f-f_0) + \delta(f+f_0)\right]$$

$$\Big\downarrow \mathcal{F}^{-1} \qquad = A\cos(\phi)\left[\frac{\delta(f-f_0) + \delta(f+f_0)}{2}\right]$$

$$\boxed{y(t) = A\cos(\phi)\cos(2\pi f_0 t)}$$

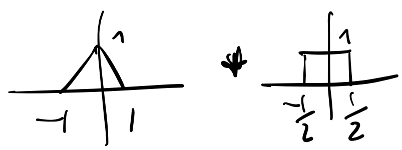
ESERCIZIO 3

Sia $x(t) = sinc^{26}(t)$

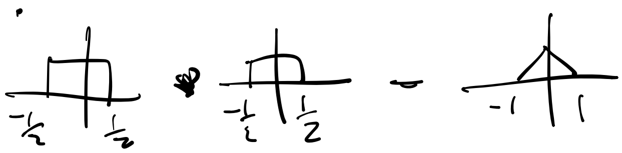
a) Determinare il supporto di $X(f)$ / banda del segnale

$$X(f) = \underbrace{\boxed{rect(f) * rect(f)} * \cdots * rect(f)}_{26 \text{ volte}}$$

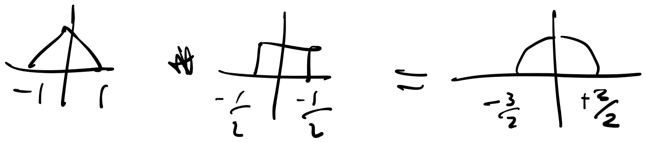
$$rect(f) * rect(f) = tri(f)$$



Supporto è dato dalla somma dei supporti dei singoli segnali.



$$\left[-\frac{1}{2}-\frac{1}{2}, \frac{1}{2}+\frac{1}{2}\right] = [-1, 1]$$



$[-1-\frac{1}{2},\ 1+\frac{1}{2}] = [-\frac{3}{2}, \frac{3}{2}]$

supporto di $X(f)$ è $[-\frac{1}{2}\cdot 26,\ \frac{1}{2}\cdot 26] = [-13, 13]$

b) $f_{s,min}$ per campionamento e ricostruzione perfetta

Banda $B=13$

$$f_{s,min} = 2B = \boxed{26}$$

c) Posso prendere $f_s = 2B = 26$ perché è convoluzione di rettangoli e si può mostrare che

$$\boxed{X(\pm 13) = 0}$$

le repliche si toccano solo nei punti dove lo spettro vale 0.

Esercizio 4

Dato

$$x(t) = \cos(2\pi t)$$

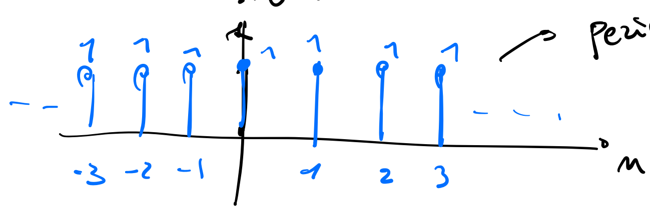
a) Campionare il segnale con $T_s=2$ e mostrare che la sequenza $x[n]$ è periodica

$$x[n] = x(nT_s) = \cos(2\pi n T_s) \qquad n \in \mathbb{Z}$$

$$= \cos(2\pi n 2) = \cos(4\pi n)$$

$$= \boxed{1}$$

$T_s = 2$



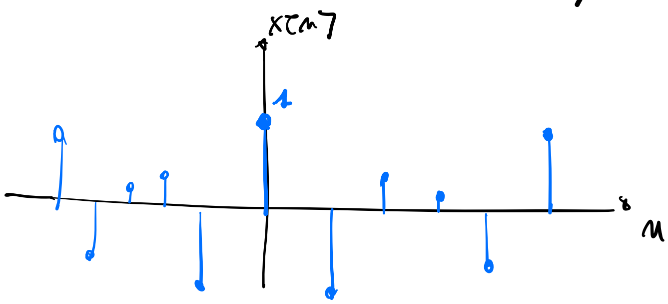
periodica di periodo $\boxed{1}$

b) $T_s = \sqrt{2}$

$$x[n] = x(nT_s)$$

$$= \cos(2\pi n T_s) \qquad T_s = \sqrt{2}$$

$$= \cos(2\pi \sqrt{2}\, n)$$



È periodica ?

$\exists N > 0$ dove N intero tale che $x[n+N] = x[n] \ \forall n \in \mathbb{Z}$

$$\cos(2\pi(n+N)\sqrt{2}) = \cos(\underbrace{2\pi n \sqrt{2}}_{x[n]})$$

$$\cos(2\pi n\sqrt{2}\, n + 2\pi N\sqrt{2})$$

$$\cos(\theta + 2m\pi) = \cos(\theta)$$

$\cancel{2\pi} N \sqrt{2} = \cancel{2} m \cancel{\pi}$ per qualche $m \in \mathbb{Z}$

$\sqrt{2} N = m \qquad N \in \mathbb{N}$ e $m \in \mathbb{Z}$

irrazionale $\Rightarrow \nexists m$

c) In generale

$$x(t) = \cos(2\pi f_0 t + \phi)$$

$$\downarrow$$

$$x[n] = x(nT_S) = \cos(2\pi \boxed{f_0 T_S} n + \phi)$$

$\hookrightarrow f_0 T_S = \dfrac{f_0}{f_S} \in \mathbb{Q}$

RAPPORTO RAZIONALE