ESERCITAZIONE 6

NOTAZIONE

QUANTIZZAZIONE

Quantizzazione uniforme in $[-V, V]$ con $L = 2^b$ livelli.

$$\Delta = \frac{2V}{L} \qquad P_{eq} \approx \frac{\Delta^2}{12}$$

pot. errore di quantizzazione

Supponendo che le ampiezze del segnale tra $[-V, V]$ siano uniformi. $P_x \approx \frac{V^2}{3}$

In decibel

$$P_{dB} = 10 \log_{10}(P)$$

Il rapporto segnale-rumore

$$SNR_{dB} = 10 \log_{10}\left(\frac{P_x}{P_{eq}}\right)$$

ENTROPIA

Entropia di una sorgente discreta:

$$H = -\sum_i p_i \log_2 p_i$$

prob. dei simboli

Esercizio 1

Si consideri un segnale analogico vocale con banda in [300, 3400] (Hz). Il segnale viene campionato e poi quantizzato uniformemente.

Assumiamo $f_s = 8$ kHz (f. campionamento)

Ampiezze normalizzate in $[-1, 1]$.

a) $f_s \geq 2B = 2 \cdot 3400 = 6.8$ kHz rispetta la condizione Shannon-Nyquist.

b) $$\Delta = \frac{\text{dinamica del segnale}}{L} = \frac{\overset{|[-1,\,1]|}{2}}{L}$$

sapendo che $L = 2^b$ ⤳ # bit di quantizzazione

$$\Delta = \frac{2}{2^b} = 2^{1-b}$$

c) Usando il fatto che $P_{eq} = \frac{\Delta^2}{12}$ determinare il # bit minimo b tale che

$$P_{dB} \leq -20 \text{ dB}$$

$$P_{dB} = 10 \log 10 \left(P_{eq} \right) = 10 \log_{10} \left(\frac{\Delta^2}{12} \right) \leq -20$$

$$\log_{10} \left(\frac{\Delta^2}{12} \right) \leq -2 \Rightarrow \quad \frac{\Delta^2}{12} \leq 10^{-2} \Rightarrow \left(2^{1-b}\right)^2 \leq 12 \cdot 10^{-2}$$

$$2^{2-2b} \le \frac{12}{100} \quad \rightarrow \quad 2-2b \le \underbrace{\log_2\left(\frac{12}{100}\right)}_{-3.06}$$

$$2b \geq 5.06$$

$$\downarrow$$

$$b \geq 2.53 \implies$$ # minimo di bit è $\boxed{3}$.

d) Assumendo che il segnale ha campioni uniformi in $[-V, V]$.

$V = 1$

$$P_x = \frac{V^2}{3} = \boxed{\frac{1}{3}}$$

e) Presa $b=3$ e $b=8$ calcolare P_{eq}, SNR_{dB} e bit-rate.

$b=3$

$$\boxed{P_{eq}} = \frac{\Delta^2}{12} = \frac{2^{2-2b}}{12} = \frac{2^{2-6}}{12} = \frac{1}{2^4} \cdot \frac{1}{12} = \frac{1}{3 \cdot 2^6} = \boxed{\frac{1}{192}}$$

$$\boxed{SNR_{dB}} = 10 \log_{10}\left(\frac{P_x}{P_{eq}}\right) = 10 \log_{10}\left(\frac{1/3}{1/192}\right)$$

$$= 10 \log_{10}(64) \simeq \boxed{18.07 \text{ dB}}$$

bit-rate

$$\downarrow$$

$$\boxed{R_b} = f_s \cdot b = 8\,\text{kHz} \cdot 3\,\text{bit} = \boxed{24\,\text{kbit/s}}$$

$b=8$

$$\boxed{P_{eq}} = \frac{2^{2-2\cdot 8}}{12} = \frac{2^{-14}}{12} = \boxed{\frac{1}{3 \cdot 2^{16}}}$$

$$\boxed{SNR_{dB}} = 10\log_{10}\left(\frac{P_x}{P_{ep}}\right) = 10\log_{10}\left(\frac{1}{\cancel{3}}\cdot\cancel{3}\cdot 2^{16}\right)$$

$$= 10\log_{10}\left(2^{16}\right)$$

$$= 160\,\underbrace{\log_{10}(2)}_{\approx 0.3011}$$

$$\approx 48.17\ dB$$

La differenza tra

$$SNR^{(8bit)} - SNR^{(3bit)} = 48.17 - 18.07 = 30.1\ dB$$

$$30.1\ dB = 5\cdot \boxed{6.02\ dB}$$ → ogni bit migliora di 6.02 dB

f) Data una funzione $x[n] = \left(\frac{15}{16}, \frac{9}{16}, \frac{3}{16}, -\frac{3}{16}, -\frac{11}{16}, -\frac{15}{16}, -\frac{7}{16}, \frac{1}{16}\right)$

Con $b=3$ $L = 2^b = 8$ livelli

$x[n] \in [-1, 1]$ $\qquad \Delta = \frac{2}{L} = \frac{2}{8} = \boxed{\frac{1}{4}}$

LIVELLI

$\left[-1, -\frac{3}{4}\right)$, $\left[-\frac{3}{4}, -\frac{1}{2}\right)$, $\left[-\frac{1}{2}, -\frac{1}{4}\right)$, $\left[-\frac{1}{4}, 0\right)$, $\left[0, \frac{1}{4}\right)$, $\left[\frac{1}{4}, \frac{1}{2}\right)$, $\left[\frac{1}{2}, \frac{3}{4}\right)$, $\left[\frac{3}{4}, 1\right]$

$-\frac{7}{8}$ $\quad \frac{1}{4} = \frac{2}{8}$ $\quad -\frac{5}{8}$ $\quad -\frac{3}{8}$ $\quad -\frac{1}{8}$ $\quad \frac{1}{8}$ $\quad \frac{3}{8}$ $\quad \frac{5}{8}$ $\quad \frac{7}{8}$

punti medi

$$x_q[n] = \left(\frac{7}{8}, \frac{5}{8}, \frac{1}{8}, -\frac{1}{8}, -\frac{5}{8}, -\frac{7}{8}, -\frac{3}{8}, \frac{1}{8} \right)$$

$$P_{x_q} = \frac{1}{8} \sum_{n=1}^{8} x_q^2[n] = \frac{1}{8} \cdot \frac{2 \cdot 7^2 + 1 \cdot 5^2 + 3 \cdot 1^2 + 1 \cdot 3^2}{8^2}$$

$$= \frac{1}{8^3} \cdot \underbrace{\left[\overbrace{2 \cdot 49}^{98} + 50 + \overbrace{3 + 9}^{12} \right]}_{160}$$

$$= \frac{160}{8^3} = \frac{\cancel{20}}{\cancel{64}} = \boxed{\frac{5}{16}}$$

$$e[n] = x_q[n] - x[n] = \left(-\frac{1}{16}, \frac{1}{16}, -\frac{1}{16}, \frac{1}{16}, \frac{1}{16}, \frac{1}{16}, \frac{1}{16}, \frac{1}{16} \right)$$

$$P_e = \frac{1}{8} \sum_{n=1}^{8} e^2[n] = \frac{1}{8} \cdot 8 \cdot \left(\frac{1}{16} \right)^2 = \left(\frac{1}{16} \right)^2 = \boxed{\frac{1}{256}}$$

$$\boxed{P_x} = \frac{1}{8} \cdot \frac{2 \cdot 15^2 + 9^2 + 2 \cdot 3^2 + 11^2 + 7^2 + 1^2}{16^2} = \frac{1}{8} \cdot \frac{720}{256} = \frac{1}{8} \cdot \frac{45}{16}$$

$$= \boxed{\frac{45}{128}}$$

$$\boxed{SNR_{dB}} = 10 \log_{10} \left(\frac{45/128}{1/256} \right)$$

$$= 10 \log_{10} \left(\frac{45}{\cancel{128}} \cdot \cancel{256}^{2} \right) = 10 \log_{10}(90)$$

$$\approx \boxed{19.54 \, dB}$$

Quella teorica deve $\approx$ 18.07 dB

Esercizio 2

Dato il segnale

$$x(t) = \frac{3}{4}\,\mathrm{sinc}(t+1) - \frac{1}{8}\,\mathrm{sinc}(t) - \frac{3}{4}\,\mathrm{sinc}(t-1)$$

Ricordiamo che

$$\underbrace{\phi_k(t)}_{\text{base ortonormale}} = \mathrm{sinc}(t-k) \quad k \in \mathbb{Z}$$

base ortonormale dello spazio segnali a banda $[-\frac{1}{2}, \frac{1}{2}]$

a) Scrivere i coefficienti nell'espansione

$$x(t) = \sum_{k\in\mathbb{Z}} a_k \underbrace{\phi_k(t)}_{\mathrm{sinc}(t-k)}$$

$a_{-1} = \frac{3}{4}$

$a_0 = -\frac{1}{8}$ $\qquad a_k = 0 \quad k \neq \pm 1, 0$

$a_1 = -\frac{3}{4}$

b) Dinamica del segnale campionato ($f_s = 1$)

$[-\frac{3}{4}, \frac{3}{4}]$

c) Quantizzazione uniforme $L=3$ livelli.

$$\boxed{\Delta} = \frac{2 \cdot \frac{3}{4}}{3} = \frac{3}{2} \Big/ 3 = \boxed{\frac{1}{2}}$$

$$\underbrace{\left[-\frac{3}{4}, -\frac{1}{4}\right)}_{I_1}, \quad \underbrace{\left[-\frac{1}{4}, \frac{1}{4}\right)}_{I_2}, \quad \underbrace{\left[\frac{1}{4}, \frac{3}{4}\right]}_{I_3}$$

$$I_1 \to -\frac{1}{2}, \quad I_2 \to 0, \quad I_3 \to \frac{1}{2}$$

$$\underbrace{-\frac{1}{2} \qquad 0 \qquad \frac{1}{2}}_{\text{punti medi}}$$

d) Calcolare coeff. quantizzati $\hat{a}_n$

$$a_{-1} = \frac{3}{4} \longrightarrow \hat{a}_{-1} = \frac{1}{2}$$

$$a_0 = -\frac{1}{8} \longrightarrow \hat{a}_0 = 0 \qquad a_n = 0 \quad n \neq \pm 1, 0$$

$$\hat{a}_1 = -\frac{3}{4} \longrightarrow \hat{a}_1 = -\frac{1}{2}$$

e) $$\hat{x}_q(t) = \sum_{n \in \mathbb{Z}} \hat{a}_n \operatorname{sinc}(t-n)$$

$$= \frac{1}{2} \operatorname{sinc}(t+1) - \frac{1}{2} \operatorname{sinc}(t-1)$$

f) Calcolare l'errore di quantizzazione

$$\|x-\hat{x}\|^2 = \sum_{n\in\mathbb{Z}} |e_n-\hat{e}_n|^2$$

continuo Parseval discreto

$$= (e_{-1}-\hat{e}_{-1})^2 + (e_0-\hat{e}_0)^2 + (e_1-\hat{e}_1)^2$$

$$= \left(\frac{3}{4}-\frac{1}{2}\right)^2 + \left(-\frac{1}{8}-0\right)^2 + \left(-\frac{3}{4}-\left(-\frac{1}{2}\right)\right)^2$$

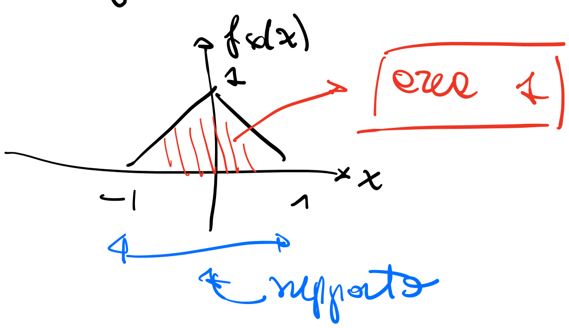
$$= 2\cdot\left(\frac{1}{4}\right)^2 + \left(\frac{1}{8}\right)^2 = 2\cdot\frac{1}{16}+\frac{1}{64} = \boxed{\frac{9}{64}}$$

$$\|x-\hat{x}\| = \boxed{\frac{3}{8}}$$

Esercizio 3

Sia X una variabile casuale con densità triangolare

$$f_X(x) = tri(x)$$



Si quantizzi X uniformemente con quant. $L=8$ su $[-1,1]$.

a) Trovare il passo di quantizzazione

$$\Delta = \frac{2}{8} = \frac{1}{4}$$

$[-1,-\frac{3}{4})$, $[-\frac{3}{4},-\frac{1}{2})$, $[-\frac{1}{2},-\frac{1}{4})$, $[-\frac{1}{4},0)$, $[0,\frac{1}{4})$, $[\frac{1}{4},\frac{1}{2})$, $[\frac{1}{2},\frac{3}{4})$, $[\frac{3}{4},1]$

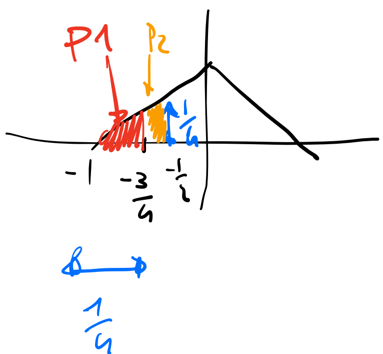
$-\frac{7}{8}$ $-\frac{5}{8}$ $-\frac{3}{8}$ $-\frac{1}{8}$ $\frac{1}{8}$ $\frac{3}{8}$ $\frac{5}{8}$ $\frac{7}{8}$

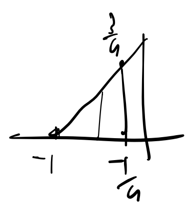
q_1 q_2 q_3 q_4 q_5 q_6 q_7 q_8

b) Calcolare le prob. di ciascun simbolo in uscita del quantizzatore

$$p_1 = \text{vedere } q_1 = \int_{-1}^{-\frac{3}{4}} (1-x)\, dx = \left[x - \frac{x^2}{2}\right]_{-1}^{-\frac{3}{4}}$$

$$= \underbrace{\frac{1}{4}}_{B} \cdot \underbrace{\frac{1}{4}}_{h} \cdot \frac{1}{2} = \frac{1}{32}$$





$$p_2 = \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2} - \frac{1}{32} = \frac{1}{8} - \frac{1}{32} = \frac{3}{32}$$

$$p_3 = \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{3}{4} - \frac{1}{8} = \frac{9}{32} - \frac{4}{32} = \frac{5}{32}$$

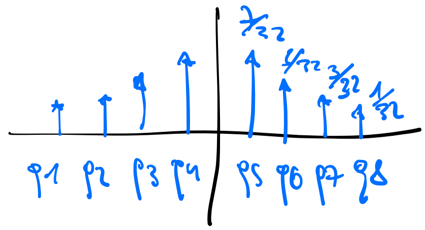
$$p_4 = \frac{1}{2} - \frac{9}{32} \Rightarrow \frac{16}{32} - \frac{9}{32} = \frac{7}{32}$$

$$p_5 = p_4 = \frac{7}{32}$$

$$p_6 = p_3 = \frac{5}{32}$$

$$p_7 = p_2 = \frac{3}{32}$$

$$p_8 = p_1 = \frac{1}{32}$$



c) Calcolare potenza dell'errore di quantizzazione

$$P_{eq} = \frac{\Delta^2}{12} = \left(\frac{1}{4}\right)^2 \cdot \frac{1}{12} = \frac{1}{16} \cdot \frac{1}{12} = \boxed{\frac{1}{192}}$$

$$\Delta = \frac{1}{4}$$

d) Calcolare l'entropia della sorgente quantizzata

$$H = -\sum_i p_i \log_2 p_i = -\frac{1}{32} \cdot \log_2 \frac{1}{32} + \dots$$

$$= \frac{1}{32} \cdot 2 \cdot \log_2 32 + \frac{3}{32} \cdot 2 \log_2 \left(\frac{32}{3}\right) +$$

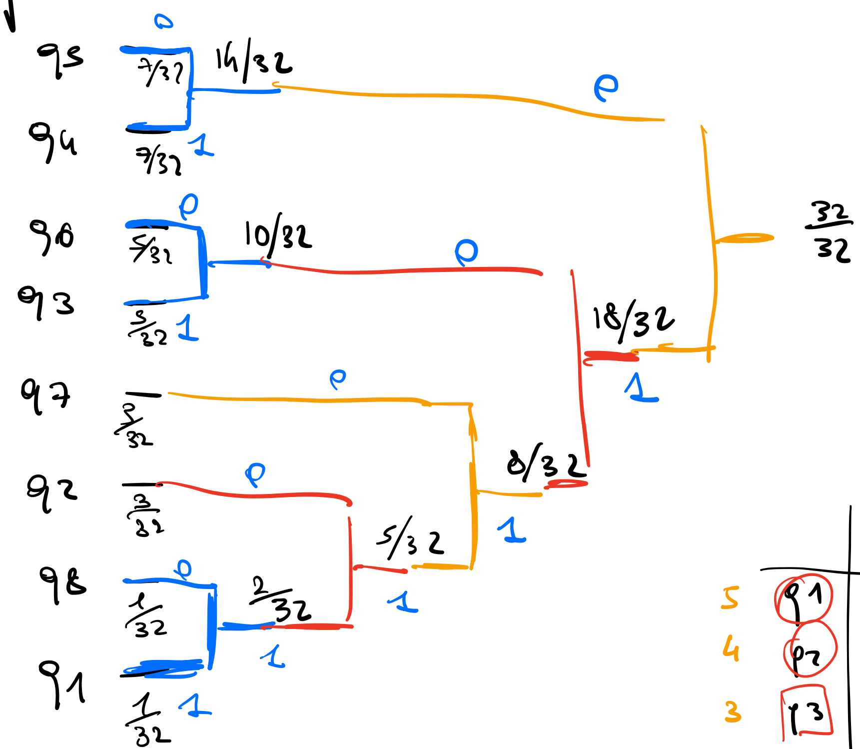
$$\frac{5}{32} \cdot 2 \log_2 \left(\frac{32}{5}\right) + \frac{7}{32} \cdot 2 \cdot \log_2 \left(\frac{32}{7}\right)$$

$\approx$ 2.749 bit/simbolo

e) Costruire un possibile codice di Huffman e calcolare la lunghezza media.

$$\bar{L} = \sum_i \underbrace{\ell_i}_{\text{lunghezza codice simbolo i-esimo}} \cdot p_i$$

Huffman



		Huffman
5	q_1	11111
4	p_2	1110
3	q_3	101
2	p_4	01
2	p_5	00
3	q_6	100
3	q_7	110
5	p_8	11110

$$\bar{L} = \underbrace{2 \cdot 5 \cdot \frac{1}{32}}_{q_1,\, q_8} + \underbrace{4 \cdot 1 \cdot \frac{3}{32}}_{q_2} + \underbrace{3 \cdot 1 \cdot \frac{3}{32}}_{q_7} + \underbrace{2 \cdot 3 \cdot \frac{5}{32}}_{q_3,\, q_6} + \underbrace{2 \cdot 2 \cdot \frac{7}{32}}_{q_4,\, q_5}$$

$$\bar{L} = \frac{10}{32} + \frac{12}{32} + \frac{9}{32} + \frac{30}{32} + \frac{28}{32} = \frac{1}{32}\left[89\right]$$

$\simeq 2.781$ bit/simbolo

$$\boxed{H \leq \bar{L} < H+1}$$ → verifica la relazione

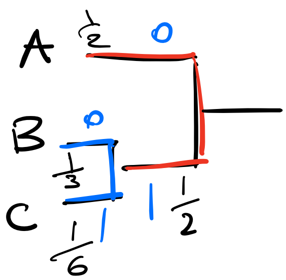
ESERCIZIO 4

Data una sorgente discreta che emette simboli (senza memoria).

$\mathcal{A} = \{A, B, C\}$ → alfabeto

$P(A) = \frac{1}{2}$, $P(B) = \frac{1}{3}$, $P(C) = \frac{1}{6}$

a) Costruire Huffman e calcolo $\bar{L}_1$ lunghezza media



	Huff
A	0
B	1 0
C	1 1

cod. prefisso

$$\bar{L}_1 = \overset{A}{1 \cdot \frac{1}{2}} + \overset{B}{2 \cdot \frac{1}{3}} + \overset{C}{2 \cdot \frac{1}{6}} = \frac{1}{2} + \frac{2}{3} + \frac{1}{3} = 1 + \frac{1}{2} = \boxed{\frac{3}{2}}$$

$\downarrow$ 1.5 bit/simboli   blocco singolo simbolo

b) Supponiamo ora la sorgente a blocchi di 2 simboli.

$$x_1 x_2 \in \mathcal{A}^2$$

$$\underbrace{\mathcal{A}^2}_{9 \text{ blocchi}} = \{ \overset{\frac{1}{4}}{AA}, \overset{\frac{1}{6}}{AB}, \overset{\frac{1}{12}}{AC}, \overset{\frac{1}{6}}{BA}, \overset{\frac{1}{9}}{BB}, \overset{\frac{1}{18}}{BC}, \overset{\frac{1}{12}}{CA}, \overset{\frac{1}{18}}{CB}, \overset{\frac{1}{36}}{CC} \}$$

c) Calcolare l'entropia della sorgente originale e quella a blocchi di 2.

originale

$$H = -\frac{1}{2}\log_2\frac{1}{2} - \frac{1}{3}\log_2\frac{1}{3} - \frac{1}{6}\log_2\frac{1}{6} \approx 1.459 \text{ bit/simbolo}$$

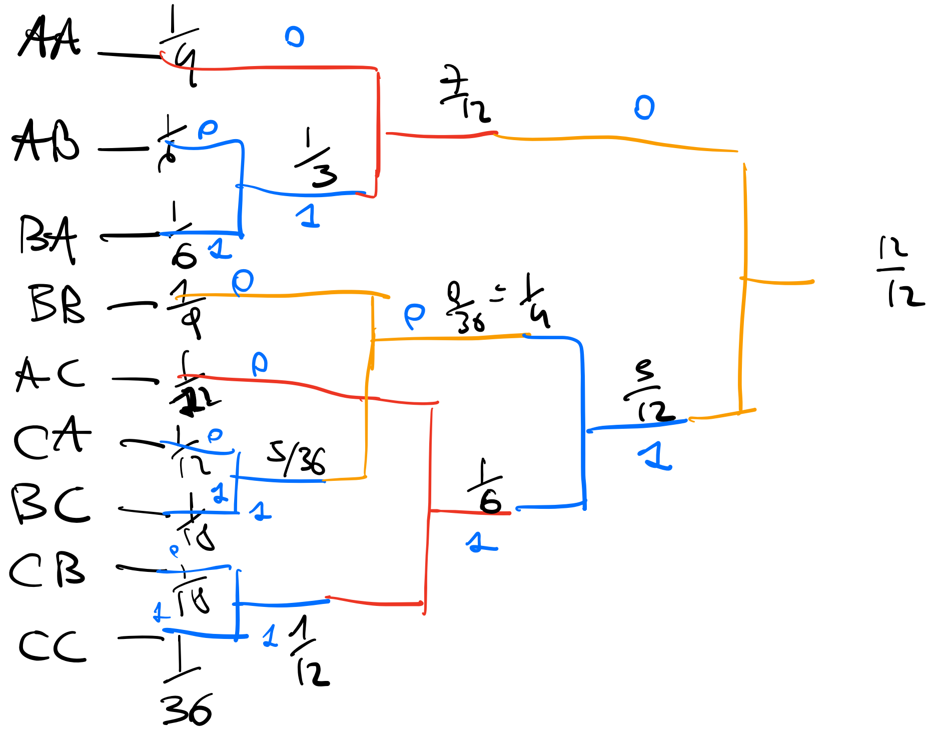
blocco 2 simboli

iid $\qquad P(x_1 x_2) = P(x_1) \cdot P(x_2)$

$$P(AA) = P(A)\,P(A) = \frac{1}{4}$$

$$\underset{\text{entropia blocco 2 simboli}}{H'} = 2\underset{\text{entropia singolo simbolo}}{H} \simeq 2.918 \text{ bit/blocco}$$

d) Codice Huffman blocco 2 simboli



	Huffman
AA	00
AB	010
AC	110
BA	011
BB	100
BC	1011
CA	1010
CB	1110
CC	1111

$$\bar{L}_2 = \overbrace{2\cdot\frac{1}{4}}^{AA} + \overbrace{2\cdot 3\cdot\frac{1}{6}}^{AB,BA} + \underbrace{3\cdot\frac{1}{9}}_{BB} + \underbrace{3\cdot\frac{1}{12}}_{AC} + \underbrace{4\cdot\frac{1}{12}}_{CA} + \underbrace{2\cdot 4\cdot\frac{1}{18}}_{BC,CB} + \underbrace{4\cdot\frac{1}{36}}_{CC}$$

$$= \frac{1}{2} + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{3} + \frac{4}{9} + \frac{1}{9}$$

$$= 1 + 1 + \frac{5}{12} + \frac{5}{9} = \frac{72+15+20}{36}$$

$$= \frac{107}{36} \sim 2.972 \text{ bit/blocco}$$

$$\frac{\overline{L_2}}{2} = \frac{2.972}{2} \sim 1.486 \text{ bit/simbolo}$$

$$< \overline{L_1} = 1.5 \text{ bit/simbolo}$$

Oss. Codificare blocchi da 2 è più conveniente in termini di $\underbrace{\text{bit/simbolo}}$

Si può mostrare che se considero blocchi di n simboli $\frac{\overline{L_n}}{n} \xrightarrow{n \to +\infty} H$ (entropia sorgente)